Stochastic partial differential equations with fractal noise

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Zusammenfassung

Die vorliegende Arbeit ist eine Studie zu stochastischen partiellen Differentialgleichungen (SPDE's). Das Ziel ist, Existenz- und Eindeutigkeitsaussagen treffen zu können für Lösungen gewisser Anfangs-Randwert-Probleme zufällig gestörter parabolischer Gleichungen bzw. Systeme auf beschränkten Gebieten des $\mathbb{R}^n$. Die zufällige Störung modelliert physikalische Einflüsse, die selbst nicht explizit bestimmbar sind, über deren grobe statistische Eigenschaften man aber Kenntnis besitzt.

Die Probleme werden im Setting konkreter zufälliger Felder formuliert. Als Rauschterme können etwa formale Raum-Zeit-Ableitungen von gebrochenen Brownschen Blättern auftreten, welche additiv oder multiplikativ in die Gleichungen eingehen. Zum Beispiel kann die Wärmeleitungsgleichung

$$\frac{\partial u}{\partial t} = \Delta u + G(u) \cdot \frac{\partial^2 B}{\partial t \partial x}$$

mit multiplikativem gebrochenen Rauschen, geeigneten Anfangsbedingungen und Dirichlet-Randdaten auf dem Einheitsintervall $(0, 1)$ betrachtet werden, B bezeichnet hier ein gebrochenes Brownsches Blatt und G eine genügend reguläre (nichtlineare) Funktion. Die Hurst-Parameter der gebrochenen Brownschen Blätter beschreiben dann nichttriviale Korrelationen der Störungen, bestimmen deren Regularität und somit auch die der Lösungen.

Letztere werden in einem geeigneten milden Sinne definiert, d.h. das Problem wird mathematisch rigoros gestellt durch eine Integralgleichung. Für obiges Besipiel ist eine (zufällige) Funktion u eine Lösung, wenn sie die Gleichung

$$u(t) = P(t)f + I_t(G(u), \frac{\partial^2}{\partial t \partial x} B)$$

erfüllt, $(P(t))_{t \geq 0}$ bezeichnet hier die zugehörige Wärmeleitungs-Halbgruppe und f die Anfangsbedingung. Die Hauptaufgabe besteht in der Formulierung eines geeigneten Integraloperators $u \mapsto I_t(G(u), \frac{\partial^2}{\partial t \partial x} B)$, mithilfe dessen der Rauschterm behandelt werden kann. Bekanntlich sind die Pfade gebrochener Brownscher Prozesse fast sicher nicht differenzierbar, und da sie im allgemeinen keinerlei Semimartingaleigenschaften besitzen, stehen in dieser Situation auch Integrale vom Itô-Typ nicht zur Verfügung.

Wir schlagen hier einen pfadweisen, d.h. zunächst komplett deterministischen Zugang vor. Techniken, die teilweise bereits Standard geworden sind bei der Behandlung gewöhnlicher stochastischer Differentialgleichungen

bezüglich der gebrochenen Brownschen Bewegung werden kombiniert mit Hilfsmitteln aus der Theorie der Funktionenräume und der Halbgruppentheorie. Die Grundidee wird in den ersten Kapiteln ausführlich illustriert und später genutzt, um einen geeigneten Integraloperator zu definieren. Die Abbildungseigenschaften des letzteren erlauben schliesslich den Beweis der gewünschten Existenz- und Eindeutigkeitsaussagen, jeweils unter vernünftigen Bedingungen an die Regularität des Rauschens. Resultate über Pfadeigenschaften der Lösungen weren parallel erhalten. Zusammen mit einfachen Hölder-Eigenschaften der Pfade gebrochener Brownscher Blätter ergeben sich dann fast sichere Aussagen für entsprechende zufällige Probleme.

Im räumlich eindimensionalen Fall bilden die diskutierten Modelle quasi gebrochene Gegenstücke bzw. Verallgemeinerungen stochastischer partieller Differentialgleichungen bezüglich Brownscher Blätter. Wie dort darf man auch hier für Raumdimensionen $n > 1$ im allgemeinen keine (zufälligen) Funktionen als Lösungen erwarten, wenn das Rauschen einer formalen Raum-Zeit-Ableitung

$$\frac{\partial^{n+1} B}{\partial t \partial x_1 \cdots \partial x_n}$$

der vollen Ordnung $n+1$ entspricht, dieser Störterm ist schlicht zu irregulär. Wir schlagen hier vor, auch Rauschterme vom Typ

$$\frac{\partial}{\partial t} \nabla B$$

zu betrachten, ∇B bezeichnet hier den räumlichen Gradienten von B. Das Rauschen ist dann partiell und räumlich gerichtet entlang der Koordinatenachsen, eine Idee, die für einige physikalische Anwendungen sinnvoll erscheint. Systeme mit solchen Rauschtermen niederer Ordnung besitzen auch in höheren Raumdimensionen Funktionenlösungen.

Insgesamt sollte dieser Zugang als ein weiterer Beitrag zur vielfältigen und keineswegs abgeschlossenen Theorie der stochastischen partiellen Differentialgleichungen gesehen werden. Durch eine vergleichsweise einfache Formulierung eröffnet er perspektivisch die Möglichkeit, auch im gebrochenen Brownschen Fall einige geometrisch und physikalisch orientierte Resultate zu erzielen, wie sie in den letzten Jahren für den Fall weissen Rauschens erarbeitet worden sind.

Contents

Chapter 1

Introduction

The central topic of this thesis is a pathwise approach to parabolic partial differential equations driven by certain random noises.
As the existing literature on the subject is quite extensive and there are several ways to define what a *stochastic partial differential equation (SPDE)* should be, it seems appropriate to mention at least some well known concepts, before we explain our own framework and emphasize what is the novelty in the stated results.
Let us start with the classical *Dirichlet problem for the heat equation in space dimension one*,

$$\frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) \ , \quad x \in (a,b),\ t \in (0,T) \ , \tag{1.1}$$

$$u(t,a) = u(t,b) = 0 \ , \quad t \in (0,T) \ , \tag{1.2}$$

$$u(0,x) = f(x) \ , \quad x \in (a,b) \ . \tag{1.3}$$

Here $(a,b) \subset (0,\infty)$ is a bounded interval, $0 < T < \infty$ some arbitrary time horizon and $f : (a,b) \to \mathbb{R}$ some suitable initial condition. Diffusion problems like (1.1)-(1.3) appear in a vast number of applications in physics, economy, the life or social sciences.
Now suppose the system under consideration is generally expected to obey (1.1)-(1.3), but there are some small scale data which themselves are unknown or too irregular to be determined explicitely and which, on the other hand, should not be neglected totally. They can be modelled introducing additional random noise terms in equation (1.1). Whenever possible, the behaviour of such noises should be chosen according to some expected or observed statistical features of the perturbation data. However, when it comes to rigorous

mathematical formulations, the different concepts of SPDE's require certain hypotheses, and that limits the freedom of choice. But the various applications and theoretical questions certainly justify the coexistence of different approaches.

Assume first that the perturbation data are of a very small scale and totally uncorrelated both in time and space. In this case, the natural choice is *space-time white noise* $\dot{W}$, the generalized centered Gaussian process on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with covariance

$$\mathbb{E}[\dot{W}(x,t)\dot{W}(y,s)] = \delta(t-s)\delta(x-y) \ ,$$

δ being Dirac's distribution at zero. More precisely, W is a random field indexed by test functions $g, h \in \mathcal{D}((0,T) \times (a,b))$ and

$$\mathbb{E}[W(g)W(h)] = \int_0^T \int_a^b f(t,x)g(t,x)dxdt \ . \tag{1.4}$$

Formally, $\dot{W}$ may be thought of as the *mixed distributional derivative*

$$\frac{\partial^2 W}{\partial t \partial x}$$

of the *Brownian sheet* on $[0,T] \times [a,b]$, the centered Gaussian process with covariance function

$$\mathbb{E}[W(t,x)W(s,y)] = (s \wedge t)(x \wedge y) \ ,$$

$s, t \in [0,T]$, $x, y \in [a,b]$. Of course, just as in the case of the other random fields we are going to consider, the realizations of the Brownian sheet are a.s. *nowhere differentiable.* For the Brownian case, this problem can be overcome using Itô's stochastic calculus.

Under *additive* white noise (the case of an external random force), equation (1.1) reads

$$\frac{\partial u}{\partial t} = \Delta u + \dot{W} \ , \tag{1.5}$$

under *linear multiplicative* white noise (the case of a random potential),

$$\frac{\partial u}{\partial t} = \Delta u + u \cdot \dot{W} \ . \tag{1.6}$$

Both (1.5) respectively (1.6) can be made precise saying that u is a *(mild) solution* if

$$u(t,x) = P(t)f(x) + \int_0^t \int_a^b p(t-s,x,y)dW(s,y) \tag{1.7}$$

respectively

$$u(t,x) = P(t)f(x) + \int_0^t \int_a^b p(t-s,x,y)u(s,y)dW(s,y) \ . \tag{1.8}$$

Here $(P(t))_{t\geq 0}$ denotes the *semigroup* on $L_2(a,b)$ associated to the *Dirichlet Laplacian* on (a,b) and $p(t,x,y)$ its *transition densities*. The stochastic integrals on the right hand sides of (1.7) and (1.8) can be defined as Itô type integrals by means of *martingale measures* (sometimes called the *'Brownian sheet approach'* or *'random field approach'*). This classical framework was presented in Walsh's lecture notes [84], Chapters two and three.

Chronologically, the cited reference was not the first important source on SPDE's, earlier papers are for instance Cabaña [13], Pardoux [70], Krylov and Rozovskij [51], Funaki [32]. See also Freidlin [31]. They mostly followed other methods for good reason: The martingale measure formulation (1.7), (1.8) breaks down for space dimensions greater than one, since only in the latter case $p(t,x,y)$ is sufficiently regular to serve as integrand for Itô type integrals. For higher dimensions, one has to turn to distribution valued concepts, cf. [84].
Probably the *infinite dimensional* approach to SPDE's has become most common. As in the deterministic case, [71], [37], one formulates equations in Hilbert (or Banach) spaces. For the above examples the canonical choice would be $L_2(a,b)$. The noise then is modelled as *Q-Wiener process*, roughly speaking a series

$$W(t) = \sum_{j=1}^{\infty} \sqrt{\lambda_j}\beta_j(t)e_j \ , \tag{1.9}$$

convergent in $L_2(a,b)$ in the mean square, where $(\beta_j(t))_{t\geq 0}$, $j = 1,2,...$ are mutually indepedent real-valued standard Brownian motions on $(\Omega, \mathcal{F}, \mathbb{P})$ and λ_j, $j = 1,2,...$ are the eigenvalues of a bounded, linear, symmetric and non-negative operator Q on $L_2(a,b)$ having finite trace $TrQ = \sum_{j=1}^{\infty} \langle Qe_j, e_j\rangle < \infty$. Here $\{e_j\}_{j=1}^{\infty}$ is an orthonormal Basis in $L_2(a,b)$. This setting can be generalized in various directions, we refer to the monograph of daPrato and Zabczyk [22]. For instance, one may drop the trace class assumption if the convergence is considered in some larger Hilbert space. In this case W is called a *cylindrical Wiener process*. Problems like (1.5) and (1.6) lead to SDE's in Hilbert spaces, of form

$$du(t) = \Delta u(t) + dW(t) \ , \quad t \in [0,T] \ , \tag{1.10}$$

or

$$du(t) = \Delta u(t) + u(t)dW(t) \ , \quad t \in [0,T] \ , \tag{1.11}$$

respectively. Solutions to (1.10) and (1.11) may for instance be formulated in the mild (evolution) sense as

$$u(t) = P(t)f + \int_0^t P(t-s)dW(s) \tag{1.12}$$

and

$$u(t) = P(t)f + \int_0^t P(t-s)u(s)dW(s) \ , \tag{1.13}$$

respectively, but also other notions of solution (strong, weak, variational) are customary. The stochastic integrals in (1.12) and (1.13) are explained as *stochastic convolutions* based on Itô type integrals with respect to the Q- (or cylindrical) Wiener process W. We omit further details and refer to [70], [32], [74], [22], [35] and [72]. From the point of view of applications, this infinite dimensional approach is much more flexible than the martingale measure approach.

Without mentioning we have coloured the noise in space. That is, encoded in the eigenvalues λ_j of Q, we have introduced some spatial correlations which determine the properties of W and consequently that of the solution process. Non-trivial spatial correlations (i.e. not just of delta type) describe the case that for two different points in space the noise data may exhibit dependencies. Very rough Q-noises W may lead to solutions whose paths are no longer locally integrable functions of the space variable (for fixed time), but proper distributions.

In space dimension one, it is possible to pass from the infinite dimensional formulation (1.10), (1.11) in the Hilbert space $L_2(a,b)$ to the finite dimensional (1.5), (1.6) with space-time white noise. This means to evaluate the 'a priori forbidden' case $Q = id$, and to make this idea rigorous, it is necessary

to integrate once more with respect to the space variable, see [22], Section 4.3.3. Alternatively, one can use certain regularization techniques, see e.g. the papers by Bertini and Cancrini [12] or Hu and Nualart [44].
On the other hand, knowingly introducing suitable spatial correlations in the finite dimensional model, one may characterize situations where analogues of (1.5), (1.6) or related hyperbolic equations possess function solutions even in space dimension two. We refer to the papers of Dalang and Frangos, [19], [18]. Instead of (1.4), the centered Gaussian random fields considered there have covariance structure

$$\mathbb{E}[F(g)F(h)] = \int_0^T \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(t,x)\varphi(|x-y|_2)g(t,x)dxdydt$$

for $g, h \in \mathcal{D}([0,T] \times \mathbb{R}^2)$ with some suitable function φ. $|\cdot|_2$ denotes the Euclidean norm in $\mathbb{R}^2$.
A related, very interesting paper is [30]. There the Laplacian was replaced by some other Markov generator, what leads to function solutions in some cases.

Another, completely different point of view was adopted in the book of Holden, Øksendal, Ubøe and Zhang, [42] and related papers, where *white noise calculus* was employed, [38]. Central techniques for this concept are chaos expansions, distributions in the sense of Hida and products defined in the Wick sense. This framework is a priori distribution-valued and allows to work with noises that are white both in time and space.

Now one might also want wo consider *noise correlations in time* to describe noise data whose present values take into account their future or past. In this case, classical results on Gaussian processes, cf. Kahane [47], as well as known integration concepts suggest to deal with fractional Brownian processes. A classical *fractional Brownian motion* $B^H = (B^H(t))_{t\geq 0}$ on the non-negative half-axis with Hurst index $H \in (0,1)$ is the centered Gaussian process with covariance function

$$\mathbb{E}[B^H(s)B^H(t)] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}) \ , \quad s,t \geq 0 \ . \tag{1.14}$$

This process was first considered in Kolmogorov [50]. Fractional Brownian motion is stationary and self-similar but in general it does not have independent increments. The combination of these properties make this process

interesting for many applications in engineering, finance, biology or physics, see for example [63], [73]. In fact, the desription of correlation phenomena in time series motivated its investigation, see Hurst [45] and Mandelbrot/van Ness [55].
Also the fractional Brownian motion is a.s. nowhere differentiable, but it is known to be a.s. H'-Hölder continuous for any $0 < H' < H$. The standard Brownian motion appears for $H = 1/2$, the only case where the increments are independent, the process is a (semi)martingale and classical Itô calculus applies. However, by the Gaussian property one can for instance use *Skorohod integration* in the general case, Alós/Mazet/Nualart [2], Alós/Nualart [3], Decreusefond/Üstünel [23], Coutin/Qian [14]. See also Duncan/Hu/Pasik-Duncan [25]. Alternatively, the Hölder regularity of its paths allows to define *Stieltjes type integrals*, we mention in particular Young [90], Russo/Vallois [76], Lyons [54] and Zähle [91]. In Zähle [92], Nualart/Rascanu [66], Klingenhfer/Zähle [49], Zähle [93], the approach of [91] had been exploited to solve related stochastic differential equations (SDE's). Meanwhile, subsequent papers have considered several properties of solution processes, see e.g. Nualart/Sausserau [67]. A comprehensive treatment of these topics is given in Mishura's lecture notes [58].
Studies of parabolic SPDE's involving fractional Brownian noises had been presented for example in Grecksch/Anh [34], Duncan/Maslowski/Pasik-Duncan [26], Tindel/Tudor/Viens [80], Maslowski/Nualart [56], Duncan/Maslowski/Pasik-Duncan [27] and recently, Maslowski/Pospisil [57]. These references follow the infinite dimensional approach with *Q- or cylindrical fractional Brownian motions*, i.e. noises of type

$$B^H(t) = \sum_{j=1}^{\infty} \sqrt{\lambda_j}\beta_j^H(t)e_j$$

analogous to (1.9) but with $(\beta^H(t))_{t\geq 0}$ being mutually independent fractional Brownian motions with Hurst exponent $H \in (0,1)$. Again the spatial correlations are encoded in the choice of the covariance operator Q.
The white noise methods of [42] were extended to the fractional case in Øksendal/Zhang [68].
Hu and Nualart [44] study different formulations of heat equations under multiplicative noises that are fractional in time and white in space.

The topic of this thesis is the following: In a formulation close to the random field approach, we wish to study reasonable *counterparts* of (1.5) and (1.6), but with noises that are of *fractional Brownian type both in time and space.* More precisely, they arise as the formal derivatives

$$\frac{\partial^2 B^{\alpha,\beta}}{\partial t \partial x}$$

of *anisotropic fractional Brownian sheets* $B^{\alpha,\beta} = \{B^{\alpha,\beta}(t,x) : t \in [0,T], x \in \mathbb{R}\}$, which are the centered Gaussian random fields on $[0,T] \times \mathbb{R}$ having the covariance function

$$\mathbb{E}[B^{\alpha,\beta}(t,x)B^{\alpha,\beta}(s,y)] = \frac{1}{2}(s^{2\alpha} + t^{2\alpha} - |t-s|^{2\alpha})\frac{1}{2}(|x|^{2\beta} + |y|^{2\beta} - |x-y|^{2\beta}) \ . \tag{1.15}$$

Here $\alpha \in (0,1)$ and $\beta \in (0,1)$ denote the temporal and spatial Hurst index, respectively. These multiparameter processes have first been considered in Kamont [48], see also Ayache [5], Ayache/Leger/Pontier [6], Ayache/Xiao [7]. They are natural generalizations of the fractional Brownian motion and have similar properties (in some sense). The Brownian sheet appears as the special case $\alpha = \beta = \frac{1}{2}$. To be consistent with our notation in the main text, we consider $B^{1-\alpha,1-\beta}$, $\alpha, \beta \in (0,1)$ in the following discussion instead of $B^{\alpha,\beta}$.

In view of the mentioned absence of martingale properties, the main issue is to introduce suitable tools to replace the Itô type integrals in (1.7) and (1.8).

The first results on parabolic equations driven by anisotropic fractional Brownian noises are due to Y. Hu, see [43] and the references cited there. Partially following the approach of [25], the paper [43] uses *Skorohod type integrals* to obtain solutions to global Cauchy problems for the corresponding analogues of equation (1.6) in arbitrary space dimensions, provided all Hurst indices are greater than 1/2. For instance, the one-dimensional (formal) equation

$$\frac{\partial u(t,x)}{\partial t} = \Delta u(t,x) + u(t,x) \cdot \frac{\partial^2 B^{1-\alpha,1-\beta}}{\partial t \partial x} \quad , \ s > 0 \ , \ x \in \mathbb{R} \ , \tag{1.16}$$

along with initial condition (1.3), is said to have the solution u if some measurability and integrability conditions hold and

$$u(t,x) = P(t)f(x) + \int_0^t \int_{\mathbb{R}} p(t-s,x,y)u(s,y)dB^{1-\alpha,1-\beta}(s,z) \ , \ s > 0 \ , \ x \in \mathbb{R} \ . \tag{1.17}$$

The stochastic integral on the right-hand side is of Skorohod type, given by the formal chaos expansion

$$\sum_{m=1}^{\infty} I_m(f_m(t,x)) \ . \tag{1.18}$$

Here $I_m(f_m(t,x))$ denotes the m-th iterated integral of a suitably symmetrized version of

$$\int_{\mathbb{R}} p(t-s_m, x-x_m)\cdots p(s_2-s_1, x_2-x_1)p(s_1, x_1-y)f(y)dy \ ,$$

$p(t,x)$ denoting the usual Gaussian heat kernel. See Nualart [64] for some background. Iteratively, the sum (1.18) is shown to converge, and this implies the existence of the respective unique solution. In particular, one obtains function solutions in dimension one, provided $1-\alpha$ and $1-\beta$ are greater than $1/2$. For general n-dimensional analogues involving n-parameter anisotropic fractional Brownian sheets (in the sense of Kamont) with temporal Hurst index $1-\alpha$ and spatial Hurst indices $1-\beta_1$,...,$1-\beta_n$, function solutions exist if $1-\alpha, 1-\beta_1, ..., 1-\beta_n > 1/2$ and $n < \sum_{i=1}^{n}(1-\beta_i)+2/(1-2\alpha)$. The latter condition is always fulfilled for $n \leq 4$, a consequence of the fact that the stochastic integral term formally leads to a product definition in the Wick sense. If this dimension condition does not hold, the solutions are members of some Watanabe type test function space based on $L_2(\Omega, \mathcal{F}, \mathbb{P})$.
[43] contains statements concerning the long time behaviour of the solutions. Results on path regularity are not discussed. The method is not so well suited for models with non-linear multiplicative noise terms.

Our approach to such equations is different. We consider parabolic Dirichlet problems on *bounded* domains, *first purely deterministic and then stochastic in the pathwise sense.* We introduce a suitable integral operator I_t, $t \in [0,T]$, such that (mild) solutions of formal equations like

$$\frac{\partial u}{\partial t} = \Delta u + \frac{\partial^2 B^{1-\alpha,1-\beta}}{\partial t \partial x} \tag{1.19}$$

or

$$\frac{\partial u}{\partial t} = \Delta u + u \cdot \frac{\partial^2 B^{1-\alpha,1-\beta}}{\partial t \partial x} \tag{1.20}$$

together with initial and boundary conditions (1.2), (1.3) are given by

$$u(t) = P(t)f + I_t(\frac{\partial^2}{\partial t \partial x} B^{1-\alpha,1-\beta}) \tag{1.21}$$

or

$$u(t) = P(t)f + I_t(u, \frac{\partial^2}{\partial t \partial x} B^{1-\alpha,1-\beta}) \tag{1.22}$$

respectively. Here $I_t(\frac{\partial^2}{\partial t \partial x} B^{1-\alpha,1-\beta}) := I_t(1, \frac{\partial^2}{\partial t \partial x} B^{1-\alpha,1-\beta})$.
To define such operators, we employ methods from [76], [91], [92], [66], [56] and use *Stieltjes type integrals*. Necessary tools are *semigroup theory*, *the theory of function spaces* and in particular *the definition of pointwise products* of functions and distributions *by means of Fourier analysis*. We prove *existence and uniqueness* of solutions to a large class of Dirichlet problems for parabolic PDE's under the influence of additive or multiplicative fractional noise terms, including (1.19) and (1.20). Non-linearities may occur.

This approach was developed in the papers in the papers [40] and [41]. The content of the present thesis essentially coincides with that of these papers, however, the exposition here is a bit more detailed.

Roughly speaking, the following results are obtained: In order to guarantee the a.s. existence of a function solution u to equation (1.19) together with (1.2) and (1.3), we need $\alpha + \beta/2 < 1$, in terms of the Hurst indices $1-\alpha$, $1-\beta$, this means $2(1-\alpha) + (1-\beta) > 1$. The spatial regularity δ of u is described in terms of a suitable Sobolev space and γ denotes its Hölder regularity in time, evaluated in the norm of that Sobolev space. The possible values must satisfy $\gamma + \alpha < 1$ and $2\gamma + \delta < 2 - 2\alpha - \beta$, and the spatial smoothness of the initial condition f needs to be slightly bigger than $2\gamma + \delta < 3/2$ (a condition caused by the analytical properties of the boundary value problem in L_2, not by the pathwise integral).
For the analogous Dirichlet problem associated with (1.20), we measure also the temporal smoothness γ of the solution in some kind of Sobolev norm, cf. [56]. The initial condition f has to be of the same quality as in the linear case. Then a function solution u with spatial smoothness δ and temporal smoothness γ a.s. exists if $\alpha < \gamma < 1-\alpha$, $2\gamma + \delta < 2 - 2\alpha - \beta$ and $\delta > \beta$. It is a.s. unique in a hybrid function space encoding these regularity properties. Note that in particular, we need a temporal Hurst index $1-\alpha > 1/2$ of the driving field. However, if $1-\alpha$ is large enough, we may allow for $1-\beta \leq 1/2$.

This differs slightly from the hypotheses for the one-dimensional case in [43]. Also non-linear problems are considered, they require additional restrictions that are familiar from other papers, [56], [80], [36]. For more precise statements we refer to the main text, especially to Chapter 4.
In this thesis, we do not address global Cauchy problems, since for equation (1.20) the behaviour of the fractional Brownian sheets at infinity does not permit a simple contraction principle. However, for (1.19) global Cauchy problems could be studied easily, using suitable weighted function spaces.
Also, we restrict attention to Dirichlet boundary conditions, since to pass to Neumann or mixed boundaries requires a modification of the analytical ingredients rather than of the pathwise integration method.
Finally, we do not study the long time behaviour here.

Our results are formulated in terms of a simple generalization: We consider *systems of equations under gradient type noises*, i.e. noises arising as the formal derivatives

$$\frac{\partial}{\partial t}\nabla B^{1-\alpha,1-\beta}$$

of suitable random fields $B^{1-\alpha,1-\beta} = \left\{B^{1-\alpha,1-\beta}(t,x) : t \in [0,T], x \in \mathbb{R}^n\right\}$ of fractional Brownian type, obtained by taking the spatial parameter x to be from $\mathbb{R}^n$ and taking the Euclidean norm on $x \in \mathbb{R}^n$ in (1.15), see [53] for comparison. Noises of this type are *directed in space*, what of course leads to models very different from the classical ones (except in space dimension one). The integration, respectively differentiation then is partial, and function solutions can be obtained in arbitrary space dimensions. The spatially one-dimensional examples discussed above are contained as special cases for $n = 1$.
Formal gradients of random fields on Euclidean spaces have already been considered some time ago. They exhibit interesting geometric features, and in the stationary isotropic case they are related to simple models in classical turbulence theory. See Dudley [24], Itô [46], Monin/Yaglom [60] and Wong/Zakai [86]. We expect this formulation to be interesting for a number of applications. For noisy non-linear evolution equations involving gradients of the solution fields, see for instance Benth/Deck/Potthoff/Streit [10].

To summarize, the new aspects of our approach to this type of equations are the *pathwise method*, the consideration of *boundary initial value problems*, relatively strong *regularity results*, the possibility to consider *non-linearities* in this model and the formulation involving *gradients of random fields*.

We proceed as follows: In the second chapter, which follows this introduction, we collect some tools and preliminaries, in particular elements of fractional calculus in Banach spaces together with some facts from semigroup theory and the theory of function spaces. Also, we state definitions and properties of fractional Brownian random fields that serve as prototypes for the driving.
In the third chapter the Stieltjes type (forward) integrals of [91] are considered in a Banach space-valued setting and the connection between the forward integrals of [76] and some Sobolev spaces is outlined. As an illustration, these methods are combined to obtain a Stieltjes type integral over space-time. Relations to stochastic Itô type integrals are discussed. These results are rather simple, therefore their short proofs are given in between. However, they mark the point of view that leads to what follows.
Namely, to the results presented in the fourth chapter, which propose a purely pathwise approach to some parabolic PDE's under random perturbations: The discussion starts with the definition of an appropriate integral operator. In fact, it is an analog of the space-time Stieltjes type integral sketched before. This allows to pose related boundary initial value problems for systems of parabolic PDE's with additive or multiplicative perturbations in a rigorous manner. Having specified these, we can state results on existence and uniqueness. Regularity properties of the solutions are part of the statements. Also related non-linear problems are discussed.
These results, so far deterministic, are then applied to the paths of the fractional Brownian random fields from the second chapter.
Since they are rather long and contain some technicalities, the proofs of the results on PDE's are shifted to the last part of the text. The key step is to establish mapping properties of the integral operator within suitable function spaces and to show it is well defined in the respective situation. For linear systems under additive noise, existence and uniqueness of solutions are then immediate, for the other cases these results follow by standard fixed point arguments.

To conclude this introduction, let us come back to the Brownian case and recall (1.5) and (1.6). Such finite dimensional (one-dimensional) SPDE's involving space-time white noise have turned out to be highly interesting: The discrete counterpart of (1.6) is a central object in the theory of ran-

dom media, cf. Anderson, Carmona and Molchanov [4], [16], [59]. Solutions to equation (1.6) itself have some intermittence properties, [12]. The existence and quality of densities of solutions for instance have been investigated in Bally/Pardoux [9] and Nualart [64]. Hitting probabilities and Hausdorff dimensions were studied in Mueller/Tribe [61], Wu/Xiao [87], Dalang/Khoshnevisan/Nualart [21]. Parabolic SPDE's as natural limits of discrete schemes have already been considered in [32]. Variants of (1.6) permit to access some non-linear equations such as the Burgers equation.
The author is interested in similar questions for equations driven by fractional Brownian noises and the analogies and differences that might occur. But these topics are not discussed in the present thesis.
Also some other questions remain to be answered. For instance, what is the precise relation between our pathwise formulations and those involving Skorohod integration with Hurst indices greater than $1/2$ (in particular, when do trace corrections behave nicely under iteration) ? Can one combine regularization and Feynman-Kac techniques with the pathwise method to obtain solutions to global problems with linear multiplicative noise ?

Chapter 2

Tools and preliminaries

This chapter contains a survey on some notions, results and ways of notation that are used in the main chapters below.

2.0.1 General notation

Let $k, n \in \mathbb{N} \setminus \{0\}$. $\{e_1, ..., e_n\}$ denotes the standard basis and $|\cdot|_n$ the Euclidean norm in $\mathbb{R}^n$, n is suppressed from notation if $n = 1$.

Given a normed vector space $(E, \|\cdot\|_E)$, the k-fold product space $\prod_{j=1}^k E$ is endowed with the l_1-norm $\sum_{j=1}^k \|f_j\|_E$, $f = (f_1, ..., f_k) \in \prod_{j=1}^k E$.

$\mathcal{M}(n \times k, \mathbb{R})$ denotes the space of real $(n \times k)$-matrices. For two members $B = (b_l^j)_{\substack{l=1,...,n \\ j=1,...,k}}$ and $C = (c_l^j)_{\substack{l=1,...,n \\ j=1,...,k}}$ of $\mathcal{M}(n \times k, \mathbb{R})$, with row vectors $b_l = (b_l^1, ..., b_l^k)$, $c_l = (c_l^1, ..., c_l^k)$, $l = 1, ..., n$, and column vectors $b^j = (b_1^j, ..., b_n^j)$, $c^j = (c_1^j, ..., c_n^j)$, $j = 1, ..., k$, we use the notation

$$\langle B, C \rangle := \left(\left\langle b^1, c^1 \right\rangle, ..., \left\langle b^k, c^k \right\rangle\right) , \tag{2.1}$$

where each component of the real k-vector on the right hand side is given by the standard scalar product on $\mathbb{R}^n$, $\langle b^j, c^j \rangle = \sum_{l=1}^n b_l^j c_l^j$. Obviously $\langle B, C \rangle = \sum_{l=1}^n b_l \cdot c_l$, with summation in $\mathbb{R}^k$, where

$$b_l \cdot c_l := (b_l^1 c_l^1, ..., b_l^k c_l^k) , \tag{2.2}$$

a notation we will prefer at some occasions later on.
We do not write the transposition of vectors explicitely, it will always be apparent from the context.

Positive constants whose values are not of importance are denoted by c, their values may differ from one occurence to the other.

2.1 Fractional calculus

In a vector-valued setting, we consider some definitions from fractional calculus that have become standard in the scalar-valued case.
Let X be a separable complex Banach space with norm $\|\cdot\|_X$ and let I be an interval or $\mathbb{R}$. In later applications, we will restrict the attention to real subspaces of X.
For $1 \leq p < \infty$ let $L_p(I, X)$ denote the space of (equivalence classes of) measurable functions $f : I \to X$ such that

$$\|f|L_p(I, X)\| = \left(\int_I \|f(t)\|_X^p \, dt\right)^{1/p} < \infty ,$$

the integrals taken in the sense of Bochner, see e.g. [39]. We write $L_1(I)$ in case $X = \mathbb{C}$.
All proofs of the facts we quote in the following carry over from the scalar valued case, we refer the reader to [77].

Let $I = (a, b)$ be a bounded interval. Given $\alpha > 0$ and a function $\varphi \in L_1((a, b), X)$, consider the *(left- and right-sided) Riemann-Liouville fractional integrals of order α of the function φ*, given by

$$I^\alpha_{a+}\varphi(t) := \frac{1}{\Gamma(\alpha)} \int_a^t \frac{\varphi(\tau)}{(t-\tau)^{1-\alpha}} d\tau$$

and

$$I^\alpha_{b-}\varphi(t) := \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_t^b \frac{\varphi(\tau)}{(\tau-t)^{1-\alpha}} d\tau \ .$$

Here for $\alpha > 0$ the powers are understood as usual in the sense of choosing the main branch of the analytic function z^α, $z \in \mathbb{C}$, with the cut along the

positive half axis. In particular, $(-1)^\alpha = e^{i\alpha\pi}$.
One has

$$I^\alpha_{a+} I^\beta_{a+} = I^{\alpha+\beta}_{a+} \quad \text{if} \quad \alpha, \beta \geq 0$$

and

$$\lim_{\varepsilon \to 0} I^\varepsilon_{a+} \varphi = \varphi$$

in $L_p((a,b),X)$, provided $\varphi \in L_p((a,b),X)$, $1 \leq p < \infty$. The same is true for I^α_{b-}.
Let $I^\alpha_{a+}(L_p((a,b),X))$ denote the space of functions $f = I^\alpha_{a+}\varphi$ with $\varphi \in L_p((a,b),X)$, similarly $I^\alpha_{b-}(L_p((a,b),X))$.

For $0 < \alpha < 1$ and functions $f \in I^\alpha_{a+}(L_p((a,b),X))$, respectively $f \in I^\alpha_{b-}(L_p((a,b),X))$, consider the *(left, respectively right-sided) Weyl-Marchaud fractional derivatives of order* α *of the function* f, formally defined as

$$D^\alpha_{a+} f(t) := \frac{\mathbf{1}_{(a,b)}(t)}{\Gamma(1-\alpha)} \left(\frac{f(t)}{(t-a)^\alpha} + \alpha \int_a^t \frac{f(t) - f(\tau)}{(t-\tau)^{\alpha+1}} d\tau \right)$$

and

$$D^\alpha_{b-} f(t) := \frac{(-1)^\alpha \mathbf{1}_{(a,b)}(t)}{\Gamma(1-\alpha)} \left(\frac{f(t)}{(b-t)^\alpha} + \alpha \int_t^b \frac{f(t) - f(\tau)}{(\tau-t)^{\alpha+1}} d\tau \right) .$$

The convergence of the hypersingular integrals on the right-hand side is considered within L_p-spaces. The following explanation is formulated for D^α_{a+} only, it holds for D^α_{b-} in an analogous way.
For $\varepsilon > 0$, introduce the truncated derivative $D^\alpha_{a+,\varepsilon}$ by

$$D^\alpha_{a+,\varepsilon} f(t) := \frac{\mathbf{1}_{(a,b)}(t)}{\Gamma(1-\alpha)} \left(\frac{f(t)}{(t-a)^\alpha} + \alpha \psi_\varepsilon(t) \right) ,$$

where

$$\psi_\varepsilon(t) := \begin{cases} \int_a^{t-\varepsilon} \frac{f(t)-f(\tau)}{(t-\tau)^{\alpha+1}} d\tau & t > a+\varepsilon \\ \frac{f(t)}{\alpha} \left[\frac{1}{\varepsilon^\alpha} - \frac{1}{(t-a)^\alpha} \right] & a \leq t \leq a+\varepsilon \ . \end{cases} \tag{2.3}$$

We quote [77], Theorem 13.1, which remains valid in our case:

Lemma 2.1.1. *Let* $f = I^\alpha_{a+}\varphi$, $\varphi \in L_p((a,b),X)$, $0 < \alpha < 1$, $1 \leq p < \infty$. *Then*

$$D^\alpha_{a+} f = \lim_{\varepsilon \to 0} D^\alpha_{a+,\varepsilon} f = \varphi$$

in $L_p((a,b),X)$, *and pointwise a.e. if* $p = 1$. *Similarly for* D^α_{b-}.

Under these assumptions $I^\alpha_{a+}D^\alpha_{a+}f = f$ in $L_p((a,b),X)$, while $D^\alpha_{a+}I^\alpha_{a+}\varphi = \varphi$ is true for any $\varphi \in L_1((a,b),X)$. This may be completed in the case $\alpha = 1$ by putting $D^1_{a+}f = df/dt$ and $D^1_{b-}f = -df/dt$, taken in the strong sense, and by the identity in the case $\alpha = 0$.
The space $I^\alpha_{a+}(L_p((a,b),X))$, $1 \le p < \infty$ will be endowed with the norm

$$\left\| f | I^\alpha_{a+}(L_p((a,b),X)) \right\| := \left\| D^\alpha_{a+}f | L_p((a,b),X) \right\| \ , \tag{2.4}$$

similarly for $I^\alpha_{b-}(L_p((a,b),X))$.

Recall (2.3). The following condition is sufficient for a function to be a member of $I^\alpha_{a+}(L_p((a,b),X))$, see [77], Theorem 13.2:

Lemma 2.1.2. *Let $0 < \alpha < 1$ and $1 \le p < \infty$. If $f \in L_p((a,b),X)$ and $\lim_{\varepsilon\to 0}\psi_\varepsilon$ (with that very f) exists in $L_p((a,b),X)$, then there exists some $\varphi \in L_p((a,b),X)$ such that $f = I^\alpha_{a+}\varphi$. Similarly for I^α_{b-}.*

Also the proof of this theorem goes through in the vector-valued case.

We mention that for globally defined functions functions $\varphi : \mathbb{R} \to X$, the right-sided Riemann-Liouville fractional integrals of order $\alpha > 0$ are given by

$$I^\alpha_{-}\varphi(t) := \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_t^\infty \frac{\varphi(\tau)}{(\tau - t)^{1-\alpha}} d\tau \ . \tag{2.5}$$

These integral operators are defined for all functions $\varphi \in L_p(\mathbb{R},X)$, $1 \le p < 1/\alpha$. For $0 < \alpha < 1$ and suitable f one can further introduce the related right-sided Weyl-Marchaud derivatives,

$$D^\alpha_{-}f(t) := \frac{\alpha(-1)^\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{f(t) - f(t+\tau)}{\tau^{1+\alpha}} d\tau \ . \tag{2.6}$$

It is known that the hypersingular integrals on the right hand side converge in $L_p(\mathbb{R},X)$, provided $f = I^\alpha_{-}\varphi$ where $\varphi \in L_p(\mathbb{R},X)$ with $1 < p < 1/\alpha$. Of course analogous assertions are true for the left-sided versions.

Finally, let E and F be two separable real Banach spaces normed by $\|\cdot\|_E$, respectively $\|\cdot\|_F$, and let $L = L(E,F)$ denote the Banach space of bounded linear operators from E into F endowed with the operator norm $\|\cdot\|_L$. Using standard estimates, Fubini's theorem for Bochner integrals and the boundedness of U, one can prove *integration-by-parts formulae* as in the scalar-valued case:

$$(-1)^\alpha \int_a^b I_{b-}^\alpha \varphi(t)\psi(t)dt = \int_a^b \varphi(t) I_{a+}^\alpha \psi(t)dt \ ,$$

if $0 \le \alpha \le 1$, $p, q \ge 1$, $1/p + 1/q < 1 + \alpha$, $\psi \in L_q((a,b),E)$ and $\varphi \in L_p((a,b),L)$. As an immediate consequence,

$$(-1)^\alpha \int_a^b U(t) D_{a+}^\alpha f(t)dt = \int_a^b D_{b-}^\alpha U(t) f(t)dt \ , \tag{2.7}$$

provided $f \in I_{a+}^\alpha(L_q((a,b),E))$ and $U \in I_{b-}^\alpha(L_p((a,b),L))$, with α, p, q as above. The limiting cases for α are justified by ordinary calculus. Of course the roles of U and f may be swapped under the appropriate change of hypotheses.

2.2 Semigroups

We sketch some connections between fractional calculus and semigroup theory that are used later on. Let $(E, \|\cdot\|_E)$ be a separable complex Banach space and $L(E)$ the space of bounded linear operators on E, endowed with the operator norm. I denotes the identity operator.

2.2.1 Semigroups and fractional powers

Assume $(P(t))_{t\ge 0} \subset L(E)$ is a *C_0-semigroup of negative type on* E, i.e. $P(t)P(s) = P(t+s)$, $s, t \ge 0$, $P(0) = I$, $\lim_{t\to 0} \|P(t)f - f\|_E = 0$, $f \in E$, and in particular

$$\|P(t)\|_L \le M e^{-\mu t} \ , \quad t \ge 0 \ , \tag{2.8}$$

with some $\mu, M > 0$. Obviously it is an equibounded family of operators. Let $-A$ denote the *infinitesimal generator* of $(P(t))_{t\ge 0}$,

$$-Af = \lim_{t\to 0} \frac{1}{t}\left(P(t)f - f\right) \quad , \quad f \in dom(-A) \ ,$$

where

$$dom(-A) = \left\{ f \in E : \lim_{t \to 0} \frac{1}{t} \left(P(t)f - f \right) \text{ exists in } E \right\} .$$

$-A$ is closed linear operator whose domain $dom(-A)$ is dense in E. Its range is again contained in E. By (2.8), A is a *positive operator*, i.e. $(-\infty, 0]$ belongs to its resolvent set, and there is some constant $C \geq 0$ such that

$$\|(A - \lambda I)\|_L \leq \frac{C}{1 + |\lambda|} \quad , \quad \lambda \in (-\infty, 0] .$$

It is most common to define the *fractional powers* A^α, $0 < \alpha < 1$ of A in terms of its resolvent,

$$A^\alpha f = \frac{\sin \pi\alpha}{\pi} \int_0^\infty \lambda^{\alpha-1} A(\lambda I + A)^{-1} f \, d\lambda ,$$

see [89], [71]. The main result of [11] gives a representation for A^α in terms of the semigroup $(P(t))_{t \geq 0}$ and *characterizes its domain*: $f \in E$ belongs to $dom(A^\alpha)$, $0 < \alpha < 1$, if and only if

$$A^\alpha f = \lim_{\varepsilon \to 0} \frac{1}{\Gamma(-\alpha)} \int_\varepsilon^\infty \frac{[I - P(u)]f}{u^{\alpha+1}} \, du \tag{2.9}$$

converges strongly in E.

From now on, assume in addition that the equibounded semigroup $(P(t))_{t \geq 0}$ is *analytic*. That means there exist some $0 < \omega \leq \frac{\pi}{2}$ and an analytic extension $z \mapsto P(z)$ of $t \mapsto P(t)$ in the sector $|\arg z| \leq \omega$ such that $P(z)$ is equibounded,

$$\lim_{\substack{z \to 0 \\ |\arg z| < \omega}} P(z)f = f \quad , \quad f \in E ,$$

and $P(z_1)P(z_2) = P(z_1 + z_2)$.
In this case the following useful properties are easy to derive: For any $f \in E$, $\alpha \geq 0$ and $t > 0$, $P(t)f$ is a member of $dom(A^\alpha)$, and for any $f \in dom(A^\alpha)$,

$$P(t)A^\alpha f = A^\alpha P(t) f \quad , \ t \geq 0 . \tag{2.10}$$

Further, (2.8) implies the bounds

$$\|A^\alpha P(t)\|_L \leq M_\alpha t^{-\alpha} e^{-\mu t} \ , \quad t > 0 , \tag{2.11}$$

with some $M_\alpha > 0$, and

$$\|P(t)f - f\|_E \leq c_\alpha t^\alpha \|A^\alpha f\|_E \ , \quad t > 0 \ , \tag{2.12}$$

for $0 \leq \alpha < 1$, $f \in dom(A^\alpha)$ and with some $c_\alpha > 0$. See [71], Chapter 2.2.

The *negative fractional powers* $A^{-\alpha}$, $\alpha > 0$, admit the representation

$$A^{-\alpha} f = \frac{1}{\Gamma(\alpha)} \int_0^\infty u^{\alpha-1} P(u) f \, du \ , \tag{2.13}$$

strongly convergent for any $f \in E$.

For $0 < \alpha_1, \alpha_2 < 1$, $\alpha_1 + \alpha_2 < 1$, we observe

$$A^{\alpha_1} A^{-\alpha_1} = I \quad , \quad A^{\alpha_1} A^{\alpha_2} = A^{\alpha_1+\alpha_2} \ ,$$

and the mappings $A^{\alpha_1} : dom(A^{\alpha_1}) \to E$ as well as $A^{\alpha_1} : dom(A^{\alpha_1+\alpha_2}) \to dom(A^{\alpha_2})$ are *isomorphisms*. In the Hilbert space case, these definitions of fractional powers agree with those deduced from the spectral theorem. We refer to [81].

From the point of view of fractional calculus, (2.9) implies that for any $f \in E$, the *(right sided) Weyl-Marchaud fractional derivative* $D^\alpha_- P(\cdot)f$ *of order* $0 < \alpha < 1$ of the function $P(\cdot)f : [0, \infty) \to E$ converges at any $t > 0$ in the pointwise sense and

$$D^\alpha_- \left(P(\cdot)f\right)(t) = (-1)^\alpha A^\alpha P(t) f \ . \tag{2.14}$$

We have used $\Gamma(1-\alpha) = \alpha\Gamma(-\alpha)$ and the definition of D^α_-. Similary, we observe from (2.13) that the *(right sided) Riemann-Liouville fractional integral* $I^\alpha_- P(\cdot)f$ *of order* $0 < \alpha < 1$ of the function $P(\cdot)f : [0, \infty) \to E$ is realized as

$$I^\alpha_- \left(P(\cdot)f\right)(t) = (-1)^{-\alpha} A^{-\alpha} P(\cdot) f(t) \ . \tag{2.15}$$

2.2.2 Bounded intervals and scales of Banach spaces

Now suppose there is a *scale of Banach spaces* $\left\{(E_\delta, \|\cdot\|_{E_\delta}\right\}_{\delta_- < \delta < \delta_+}$, $\delta_- < 0 < \delta_+$. That means, the continuous embedding $E_{\delta_1} \subset E_{\delta_2}$, $\delta_- < \delta_2 \leq \delta_1 < \delta_+$ is valid, and there exists a set of linear operators $\{A_\tau\}_{0 \leq \tau < \delta_+ - \delta_-}$ such that A_τ

is an isomorphic mapping from E_δ onto $E_{\delta-\tau}$, $\delta_- < \delta - \tau \le \delta < \delta_+$, $A_0 = I$ and $A_{\tau_1}A_{\tau_2} = A_{\tau_1+\tau_2}$, $\tau_1, \tau_2 \ge 0$, $\tau_1 + \tau_2 < \delta_+ - \delta_-$. See [81].
Suppose $(P(t))_{t\ge 0}$ is an analytic semigroup of negative type on E_0 with generator A. Assume that for $0 < \kappa < 1$, we have $dom(A^{\kappa/2}) = E_\kappa$, the norms $\|\cdot\|_{E_\kappa}$ and $f \mapsto \left\|A^{\kappa/2}f\right\|_{E_0}$ are equivalent and the fractional powers $A^{\kappa/2} : E_{\kappa+\delta} \to E_\delta$ act as isomorphisms. (2.10) then allows to apply the semigroup operators to a member of any E_δ, $\delta_- < \delta < \delta_+$.

Let $0 < t < T$ and $f : [0,T] \to E_{-\beta}$, $\delta_- < -\beta < 0$ be a given function. In view of our applications we reverse time and consider the *left sided fractional Weyl-Marchaud derivative* $D^\alpha_{0+}P(t-\cdot)f(\cdot)$ *of order* $0 < \alpha < 1$ of the function $P(t-\cdot)f(\cdot) : [0,t] \to E$, formally given by

$$\begin{aligned} &D^\alpha_{0+}\left(P(t-\cdot)f(\cdot)\right)(s) \qquad (2.16)\\ &= \mathbf{1}_{(0,t)}(s)\frac{1}{\Gamma(1-\alpha)}\left(\frac{P(t-s)f(s)}{s^\alpha} + \alpha\int_0^s \frac{P(t-s)f(s) - P(t-\tau)f(\tau)}{(s-\tau)^{\alpha+1}}d\tau\right)\end{aligned}$$

for $0 < s < t$, where $c_\alpha = \alpha\Gamma(1-\alpha)^{-1}$. Recall that

$$I^\alpha_{0+}\varphi(s) = \frac{1}{\Gamma(\alpha)}\int_0^s \frac{\varphi(\tau)}{(s-\tau)^{1-\alpha}}\,d\tau$$

is the *Riemann-Liouville fractional derivative of order* $0 < \alpha < 1$ of a function φ on $(0,t)$.

For $0 < \alpha < 1$ and $\delta_- < \delta < \delta_+$, let $W^\alpha([0,T], E_\delta)$ denote the Banach space of functions $f : [0,T] \to E_\delta$ such that

$$\|f|W^\alpha([0,T],E_\delta)\| := \sup_{0\le t\le T}\left(\|f(t)\|_{E_\delta} + \int_0^t \frac{\|f(t)-f(\sigma)\|_{E_\delta}}{(t-\sigma)^{\alpha+1}}\,d\sigma\right) < \infty\ .$$

Lemma 2.2.1. *Let* $0 < \alpha < 1$ *and* $f \in W^\alpha([0,T], E_{-\beta})$, $\delta_- < -\beta < \delta_+$. *Suppose* $-\beta \le \delta < 2 - 2\alpha - \beta$ *and* $\delta < \delta_+$. *Then for any* $0 < t \le T$*, the left sided Weyl-Marchaud fractional derivative* $D^\alpha_{0+}P(t-\cdot)f(\cdot)$ *of order* α *of the function* $P(t-\cdot)f(\cdot)$ *is given by*

$$\begin{aligned} &D^\alpha_{0+}\left(P(t-\cdot)f(\cdot)\right)(s) = \mathbf{1}_{(0,t)}(s)\psi(s)\ ,\ \textit{where}\\ &\psi(s) = A^\alpha P(t-s)f(s) + c_\alpha P(t-s)\int_s^\infty u^{-\alpha-1}P(u)f(s)du\\ &\qquad + c_\alpha\int_0^s u^{-\alpha-1}P(u)P(t-s)\left[f(s)-f(s-u)\right]du\ ,\end{aligned}$$

convergent in $L_1((0,t),E_\delta)$. Moreover, there exists some $\varphi \in L_1((0,t),E_\delta)$ such that $P(t-\cdot)f(\cdot) = I^\alpha_{0+}\varphi$, and the identity

$$I^\alpha_{0+}D^\alpha_{0+}P(t-\cdot)f(\cdot) = P(t-\cdot)f(\cdot) \tag{2.17}$$

holds in $L_1((0,t),E_\delta)$.

Proof. We wish to apply Lemma 2.1.2, its hypotheses had been formulated in terms of the auxiliary function (2.3). Recall (2.16) as well as (2.14) and put $N := \|f|W^\alpha([0,T],E_{-\beta})\|$. Recall also that the semigroup operators $P(t)$ are well defined, bounded and strongly continuous both in $E_{-\beta}$ and in E_δ, since the fractional powers of A act as isomorphic mappings. We first consider the integral part of (2.3) and observe that for any $\varepsilon > 0$,

$$\begin{aligned}
&\int_0^{s-\varepsilon} \frac{P(t-s)f(s) - P(t-\tau)f(\tau)}{(s-\tau)^{\alpha+1}} d\tau \\
&= \int_\varepsilon^s \frac{P(t-s)[f(s) - P(u)f(s-u)]}{u^{\alpha+1}}\, du \\
&= P(t-s)\int_\varepsilon^s \frac{P(u)[f(s) - f(s-u)]}{u^{\alpha+1}}\, du + \int_\varepsilon^\infty \frac{[I-P(u)]P(t-s)f(s)}{u^{\alpha+1}}\, du \\
&\quad - P(t-s)\int_s^\infty \frac{[I-P(u)]f(s)}{u^{\alpha+1}}\, du,
\end{aligned}$$

so far with the integrals evaluated in the norm of $E_{-\beta}$. Now consider the right hand side of the last equality. For the last summand there, property (2.11) implies that

$$\begin{aligned}
&\int_\varepsilon^t \left\| P(t-s)\int_s^\infty u^{-\alpha-1}[I-P(u)]f(s)du \right\|_{E_\delta} ds \\
&\qquad \le c\int_\varepsilon^t (t-s)^{-(\delta+\beta)/2} \left\| \int_s^\infty u^{-\alpha-1}[I-P(u)]f(s)du \right\|_{E_{-\beta}} ds \\
&\qquad\qquad \le cN\int_0^t s^{-\alpha}(t-s)^{-(\delta+\beta)/2} ds < \infty\ .
\end{aligned}$$

Taking into account also (2.12), we obtain

$$\begin{aligned}\int_\varepsilon^t \left\| P(t-s)\int_\varepsilon^s u^{-\alpha-1}P(u)[f(s)-f(s-u)]du \right\|_{E_\delta} ds \\ \leq c\int_\varepsilon^t (t-s)^{-(\delta+\beta)/2}\int_0^s \frac{\|f(s)-f(s-u)\|_{E_{-\beta}}}{u^{\alpha+1}}\, duds \\ \leq cN\int_0^t (t-s)^{-(\delta+\beta)/2} ds < \infty\end{aligned}$$

for the first summand. From these bounds we obtain the $L_1((0,t),E_\delta)$-convergence of the discussed terms.
To treat the the middle summand, we use the following equality, which was shown in [11], section 2:

$$A^\alpha\left[\int_0^\infty P(\varepsilon w)P(t-s)f(s)q_\alpha(w)dw\right] = \int_\varepsilon^\infty \frac{[I-P(u)]P(t-s)f(s)}{u^{\alpha+1}}du\ .$$

The function q_α is defined by its Laplace transform,

$$\int_0^\infty e^{-\lambda u}q_\alpha(u)du = \lambda^{-\alpha}\int_1^\infty \frac{1-e^{-\lambda u}}{u^{\alpha+1}}du\ ,$$

$Re\,\lambda > 0$. It is a member of $L_1(0,\infty)$ and satisfies

$$\int_0^\infty q_\alpha(u)du = \Gamma(-\alpha)\ ,$$

see [11], p. 193.
As $P(t-s)f(s) \in dom(A^\nu)$ for any $\nu \geq 0$ and $0 \leq s < t$, the integral

$$\left[\int_0^\infty P(\varepsilon w)P(t-s)f(s)q_\alpha(w)dw\right]$$

is an element of $dom(A^\alpha)$. If instead it is considered as a member of $E_{-\beta}$ only, we may pull out $P(t-s)$ from under the integral sign. Hence we may conclude

$$
\begin{aligned}
&\left\| A^\alpha P(t-s)f(s) - \frac{1}{\Gamma(-\alpha)} \int_\varepsilon^\infty \frac{[I-P(u)]P(t-s)f(s)}{u^{\alpha+1}} du \right\|_{E_\delta} \\
&= \left\| A^\alpha P(t-s)f(s) - \frac{1}{\Gamma(-\alpha)} A^\alpha \left[\int_0^\infty P(\varepsilon w)P(t-s)f(s) q_\alpha(w) dw \right] \right\|_{E_\delta} \\
&\qquad = \left\| A^\alpha P(t-s) \frac{1}{\Gamma(-\alpha)} \int_0^\infty [I-P(\varepsilon w)] f(s) q_\alpha(w) dw \right\|_{E_\delta} \\
&\qquad\quad \leq c(t-s)^{-\alpha-(\delta+\beta)/2} \int_0^\infty \|[I-P(\varepsilon w)]f(s)\|_{E_{-\beta}} q_\alpha(w) dw \ .
\end{aligned}
$$

This is uniformly bounded, and the strong continuity of the semigroup yields the desired result.
Next, we discuss the remaining part of (2.3) in a similar manner:

$$
\begin{aligned}
&\int_0^\varepsilon \left\| \frac{P(t-s)f(s)}{\alpha} \left[\frac{1}{\varepsilon^\alpha} - \frac{1}{s^\alpha} \right] \right\|_{E_\delta} ds \\
&\qquad \leq \frac{cN}{\alpha} \left(\int_0^\varepsilon s^{-\alpha} (t-s)^{-(\delta+\beta)/2} ds - \frac{1}{\varepsilon^\alpha} \int_0^\varepsilon (t-s)^{-(\delta+\beta)/2} ds \right) \\
&= \frac{c\varepsilon^{1-\alpha-(\delta+\beta)/2} N}{\alpha} \left(\int_0^1 \sigma^{-\alpha} (\frac{t}{\varepsilon} - \sigma)^{-(\delta+\beta)/2} ds - \int_0^1 (\frac{t}{\varepsilon} - \sigma)^{-(\delta+\beta)/2} ds \right) \ ,
\end{aligned}
$$

which for fixed t and small enough ε, is bounded by $c\varepsilon^{1-\alpha-(\delta+\beta)/2}$, note that $1-\alpha-(\delta+\beta)/2 > 0$.
This shows the convergence of the Weyl-Marchaud derivative in $L_1((0,t), E_\delta)$. The existence of the function $\varphi \in L_1((0,t), E_\delta)$ now follows from Lemma 2.1.2. To obtain the identity (2.17), it suffices to take into account Lemma 2.1.1. □

2.3 Function spaces

We collect basic facts used in the main text. Though we will use an L_2-setting w.r.t. the spatial variable, we quote key results for $1 < p < \infty$ to indicate that some of our results carry over without too much effort, cf. Remark 4.4.2 below.

2.3.1 Potential spaces

Let $\mathcal{S}(\mathbb{R}^n,\mathbb{C}^k)$ be the space of $\mathbb{C}^k$-valued Schwartz functions and $\mathcal{S}'(\mathbb{R}^n,\mathbb{C}^k)$ the space of $\mathbb{C}^k$-valued tempered distributions on $\mathbb{R}^n$. $\hat{\varphi}$, $\hat{f}$ and $\varphi^\vee$, $f^\vee$ denote the *Fourier transform* and the *inverse Fourier transform* of $\varphi=(\varphi_1,...,\varphi_k)\in\mathcal{S}(\mathbb{R}^n,\mathbb{C}^k)$ or $f=(f_1,...,f_k)\in\mathcal{S}'(\mathbb{R}^n,\mathbb{C}^k)$,

$$\hat{\varphi}(\xi)=(2\pi)^{-n/2}\int_{\mathbb{R}^n}e^{-i\langle x,\xi\rangle}\varphi(x)dx\ ,\ \ \varphi\in\mathcal{S}(\mathbb{R}^n,\mathbb{C}^k),\ \ \xi\in\mathbb{R}^n\ .$$

Here and in the following, we suppress $\mathbb{C}^k$ from notation if $k=1$, e.g. we write $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$. Recall that $|\cdot|_n$ denotes the Euclidean norm in $\mathbb{R}^n$, we write $|\cdot|$ if $n=1$. $\langle\cdot,\cdot\rangle$ denotes the Euclidean scalar product. For $f\in\mathcal{S}'(\mathbb{R}^n,\mathbb{C}^k)$, $f^\wedge(\varphi):=f(\hat{\varphi})$, $\varphi\in\mathcal{S}(\mathbb{R}^n,\mathbb{C}^k)$.

For $1<p<\infty$ and $\alpha\in\mathbb{R}$, the *Bessel potential spaces of order* α are given by

$$H^\alpha_p(\mathbb{R}^n,\mathbb{C}^k):=\left\{f\in\mathcal{S}'(\mathbb{R}^n,\mathbb{C}^k):\left\|f|H^\alpha_p(\mathbb{R}^n,\mathbb{C}^k)\right\|<\infty\right\}\ ,$$

where

$$\left\|f|H^\alpha_p(\mathbb{R}^n,\mathbb{C}^k)\right\|:=\left\|\left((1+|\xi|_n^2)^{\alpha/2}f^\wedge\right)^\vee|L_p(\mathbb{R}^n,\mathbb{C}^k)\right\|\ . \tag{2.18}$$

Note that $H^\alpha_p(\mathbb{R}^n,\mathbb{C}^k)$ may be seen as the k-fold product space $\prod_{j=1}^k H^\alpha_p(\mathbb{R}^n)$. For $\sigma\in\mathbb{R}$, the linear operator $f\mapsto I_\sigma f:=\left((1+|\xi|_n^2)^{\sigma/2}f^\wedge\right)^\vee$ is an *isomorphism* (lifting) of $H^\alpha_p(\mathbb{R}^n,\mathbb{C}^k)$ onto $H^{\alpha-\sigma}_p(\mathbb{R}^n,\mathbb{C}^k)$. In particular, for $\sigma>0$, $(1+|\xi|_n^2)^{-\sigma/2}$ is the Fourier image of the *Bessel kernel* $G_\sigma\in L_1(\mathbb{R}^n)$ and $I_{-\sigma}f$, $f\in L_p(\mathbb{R}^n)$, $1<p<\infty$, is realized as the convolution $G^\sigma f:=G_\sigma*f$.

The dual space of $H^\alpha_p(\mathbb{R}^n)$ is $H^{-\alpha}_{p'}(\mathbb{R}^n)$, $1/p+1/p'=1$. For $u\in H^\alpha_p(\mathbb{R}^n)$ and $v\in H^{-\alpha}_{p'}(\mathbb{R}^n)$, the *dual pairing* of u and v is given by $\langle u,v\rangle$, and

$$|\langle u,v\rangle|\le c\left\|u|H^\alpha_p(\mathbb{R}^n)\right\|\left\|v|H^{-\alpha}_{p'}(\mathbb{R}^n)\right\|\ . \tag{2.19}$$

For $0<\alpha<1$,

$$\left\|f|L_p(\mathbb{R}^n,\mathbb{C}^k)\right\|+\left\|\left(\int_0^1\left(\frac{1}{t^n}\int_{|h|_n<t}|f(\cdot+h)-f(\cdot)|_k dh\right)^2\frac{dt}{t^{1+2\alpha}}\right)^{1/2}|L_p(\mathbb{R}^n)\right\| \tag{2.20}$$

determines an equivalent norm in $H_p^\alpha(\mathbb{R}^n,\mathbb{C}^k)$, cf. section 2.3 in [75] and the references cited there. For $p=2$, also the simpler expression

$$\left\|f|L_2(\mathbb{R}^n,\mathbb{C}^k)\right\| + \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x+y)-f(y)|_k^2}{|y|_n^{2\alpha+n}}\,dx\,dy\right)^{1/2}$$

defines an equivalent norm in $H_2^\alpha(\mathbb{R}^n,\mathbb{C}^k)$, $0<\alpha<1$.

If $p=2$, we use the short notation

$$\|f\|_\alpha \quad \text{for the norm} \quad \left\|f|H_2^\alpha(\mathbb{R}^n,\mathbb{C}^k)\right\| \ .$$

In particular, $\|\cdot\|_0$ denotes the norm in $L_2(\mathbb{R}^n,\mathbb{C}^k)$. Similarly, when dealing with $L_\infty(\mathbb{R}^n,\mathbb{C}^k)$, we use

$$\|f\|_\infty \quad \text{to abbreviate the norm} \ \left\|f|L_\infty(\mathbb{R}^n,\mathbb{C}^k)\right\| \ .$$

2.3.2 Partial potential spaces

For later purposes it seems convenient to introduce also another type of spaces. Recall that $\{e_1,...,e_n\}$ denotes the standard basis in $\mathbb{R}^n$. For $x=(x_1,...,x_n)$ and fixed $l=1,...,n$, write $x_l'=(x_1,...,x_{l-1},x_{l+1},...,x_n)$ for the $(n-1)$-vector obtained from x by disposing the coordinate x_l, and identify x with (x_l',x_l), $x=(x_l',x_l)$.
For $l=1,...,n$ fixed, $f\mapsto f^{\wedge_l}$ denotes the *partial Fourier transform* of f with respect to x_l, i.e.

$$f^{\wedge_l}(\xi) = (2\pi)^{-1/2}\int_{\mathbb{R}_l} e^{-ix_l\xi_l} f(\xi_l',x_l)dx_l \ , \ f\in\mathcal{S}(\mathbb{R}^n),\ \xi\in\mathbb{R} \ ,$$

where $\mathbb{R}_l := span\,\{e_l\}$. $f\mapsto f^{\vee_l}$ denotes its inverse. It follows that for a C^∞-function $m(\xi_l)$ depending only on ξ_l and being at most of polynomial growth, we always have $(m(\xi_l)f^{\wedge_l})^{\vee_l} = (m(\xi_l)f^\wedge)^\vee$, where $f\in\mathcal{S}(\mathbb{R}^n)$ or $\mathcal{S}'(\mathbb{R}^n)$. See [62].
By $H_{p,l}^\alpha(\mathbb{R}^n)$, $1<p<\infty$, $\alpha\in\mathbb{R}$, we denote the space

$$H_{p,l}^\alpha(\mathbb{R}^n) := \left\{f\in\mathcal{S}'(\mathbb{R}^n) : \left\|f|H_{p,l}^\alpha(\mathbb{R}^n)\right\| < \infty\right\} \ ,$$

where

$$\left\|f|H_{p,l}^\alpha(\mathbb{R}^n)\right\| := \left\|\left((1+\xi_l^2)^{\alpha/2}f^\wedge\right)^\vee |L_p(\mathbb{R}^n)\right\| \ . \tag{2.21}$$

For $\alpha \geq 0$, $H_p^\alpha(\mathbb{R}^n)$ is continuously embedded in $H_{p,l}^\alpha(\mathbb{R}^n)$, for $\alpha < 0$ we have the converse embedding. The space $\mathcal{S}(\mathbb{R}^n)$ is dense in both spaces.
Given $\alpha_l \in \mathbb{R}$, $l = 1, ..., n$, put $\overline{\alpha} = (\alpha_1, ..., \alpha_n)$ and

$$H_p^{\overline{\alpha}}(\mathbb{R}^n) := \bigcap_{l=1}^{n} H_{p,l}^{\alpha_j}(\mathbb{R}^n),$$

which are Banach spaces if normed by $\left\|f|H_p^{\overline{\alpha}}(\mathbb{R}^n)\right\| := \sum_{l=1}^n \left\|f|H_{p,l}^{\alpha_l}(\mathbb{R}^n)\right\|$. A treatment of spaces of this type can be found in [62].

In some application below we prefer the following notation: If $g = (g_1, ..., g_n)$ denotes an index vector consisting of numbers or mappings g_l (we are only interested in the question whether a particular g_l vanishes identically or not), set

$$H_{p,g}^\alpha(\mathbb{R}^n) := \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \left\|f|H_{p,g}^\alpha(\mathbb{R}^n)\right\| < \infty \right\} ,$$

where $\alpha > 0$ and according to the above, $\left\|f|H_{p,g}^\alpha(\mathbb{R}^n)\right\|$ is given by

$$\sum_{l:g_l \neq 0} \left\| \left((1 + \xi_l^2)^{\alpha/2} f^\wedge \right)^\vee |L_p(\mathbb{R}^n) \right\| + \sum_{l:g_l = 0} \|f|L_p(\mathbb{R}^n)\| .$$

This is just the space $H_p^{\overline{\alpha}}(\mathbb{R}^n)$ where $\overline{\alpha} = (\alpha_1, ..., \alpha_n)$ with $\alpha_l = 0$ if and only if $g_l \equiv 0$, and otherwise $\alpha_l = \alpha$, $l = 1, ..., n$.
As before, we agree to abbreviate the norm in $H_{2,g}^\alpha(\mathbb{R}^n)$ by $\|\cdot\|_{\alpha,g}$.

2.3.3 Spaces on domains

Let D be a *bounded* C^∞*-domain* in $\mathbb{R}^n$. By $\mathcal{D}(D, \mathbb{C}^k)$ or $C_0^\infty(D, \mathbb{C}^k)$ we denote the space of smooth compactly supported $\mathbb{C}^k$-valued fuctions on D, and by $\mathcal{D}'(D, \mathbb{C}^k)$ its topological dual. As before, $\mathbb{C}^k$ is suppressed from the notation if $k = 1$. For more information on the following see [81].
Given $\alpha \in \mathbb{R}$ we define the spaces

$$H_p^\alpha(D, \mathbb{C}^k) := \left\{ f \in \mathcal{D}'(D, \mathbb{C}^k) : \exists g \in H_p^\alpha(\mathbb{R}^n, \mathbb{C}^k) \text{ such that } g|_D = f \right\} ,$$

where $g|_D$ denotes the restriction in the sense of distributions. We equip them with the norm

$$\left\|f|H_p^\alpha(D, \mathbb{C}^k)\right\| := \inf \left\{ \left\|g|H_p^\alpha(\mathbb{R}^n, \mathbb{C}^k)\right\| : g \in H_p^\alpha(\mathbb{R}^n, \mathbb{C}^k) \text{ such that } g|_D = f \right\} ,$$

the infimum taken over all such g. In particular,

$$\left\| f|_D | H_p^\alpha(D,\mathbb{C}^k)\right\| \leq c \left\| f | H_p^\alpha(\mathbb{R}^n,\mathbb{C}^k)\right\|$$

for $f \in H_p^\alpha(\mathbb{R}^n,\mathbb{C}^k)$, cf. [81], 4.2.2.
The space $\mathring{H}_p^\alpha(D,\mathbb{C}^k)$ is defined as the completion of $C_0^\infty(D,\mathbb{C}^k)$ in the norm $\left\| \cdot | H_p^\alpha(D,\mathbb{C}^k)\right\|$.
One further defines the spaces

$$\widetilde{H}_p^\alpha(D,\mathbb{C}^k) := \left\{ f \in H_p^\alpha(\mathbb{R}^n,\mathbb{C}^k) : supp\, f \subset \overline{D} \right\} \ ,$$

where *supp* f denotes the support of f (in distributional sense) and

$$H_{p,0}^\alpha(D,\mathbb{C}^k) := \left\{ f \in H_p^\alpha(D,\mathbb{C}^k) : f|_{\partial D} = 0 \right\} \ ,$$

where $f|_{\partial D}$ is the restriction (trace) of f to the boundary ∂D of D, see [81], 4.7.1.
It is known that $\widetilde{H}_p^\alpha(D,\mathbb{C}^k) = \mathring{H}_p^\alpha(D,\mathbb{C}^k)$ if $-1/p < \alpha < \infty$, $\alpha - 1/p \notin \mathbb{Z}$, and that $H_p^\alpha(D,\mathbb{C}^k) = \mathring{H}_p^\alpha(D,\mathbb{C}^k)$ if $-\infty < \alpha \leq 1/p$, see [81], Section 4.3.2.
We put

$$\overline{H}_p^\alpha(D,\mathbb{C}^k) := \begin{cases} \widetilde{H}_p^\alpha(D,\mathbb{C}^k) & \text{if } \alpha \geq 0 \\ H_p^\alpha(D,\mathbb{C}^k) & \text{if } \alpha < 0 \ . \end{cases}$$

2.3.4 Differential operators

Though our method can be extended to more general situations, we restrict our attention to Dirichlet boundary initial value problems associated to some second order elliptic differential operators considered in $L_2(D)$.
Let A_0 be a self-adjoint operator in $L_2(D)$ with domain $dom(A_0) = H_{2,0}^2(D)$, obtained as the Friedrichs extension of some second order differential operator A_D,

$$(A_D f)(x) = -\sum_{i,k=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ik}(x)\frac{\partial f}{\partial x_k}(x)\right) + c(x)f(x)$$

$f \in dom(A_D) = C_0^\infty(D)$, fulfilling the ellipticity condition $\sum_{i,k} a_{ik}(x)\xi_i\xi_k \geq \lambda\left|\xi\right|^2$, $x \in D$, $\xi \in \mathbb{R}^n$ with some $\lambda > 0$, and having real-valued coefficients $a_{ik} = a_{ki} \in C^\infty(D)$, $c \in C^\infty(D)$, $c(x) \geq 0$, $x \in \mathbb{R}^n$, which, together with all their derivatives, can be extended continuously to $\overline{D}$, see e.g. [1] or [81].

Standard arguments show that A_0 is non-negative, has pure point spectrum and its eigenfunctions form an orthonormal basis of $L_2(D)$.
By the choice of the domain, *Dirichlet boundary conditions* are imposed. The simplest example is the *Dirichlet Laplacian* $-\Delta$ on $D \subset \mathbb{R}^n$.

Later we will turn to systems of differential equations. Let B be a real $(k \times k)$-matrix, such that all eigenvalues of B are contained in the half plane $\{z \in \mathbb{C} : Re\, z > 0\}$. We consider $A = BA_0$, more precisely: Given $u = (u_1, ..., u_k) \in C_0^\infty(D, \mathbb{C}^k)$, we set

$$Au := B(A_0 u_1, ..., A_0 u_k) \ , \tag{2.22}$$

with the usual matrix multiplication. We refer to B as the *cross diffusion matrix*.
From the spectral representation of A_0 it can be deduced that A is a sectorial operator, hence $-A$ generates an analytic semigroup $(P(t))_{t\geq 0}$ on $L_2(D)$. This semigroup is of negative type. A proof is carried out in [69] for $A_D = -\Delta$, but the arguments work for general A_D.
It is further shown that

$$dom(A) = \prod_{j=1}^{k} dom(A_0) \quad \text{and} \quad dom(A^\alpha) = \prod_{j=1}^{k} dom(A_0^\alpha) \ , \tag{2.23}$$

that means $domA = H^2_{2,0}(D, \mathbb{C}^k)$. On the other hand, complex interpolation shows that $domA_0^{\alpha/2}$ equals $\overset{\circ}{H}{}_2^\alpha(D)$ if $1/2 \leq \alpha < 3/2$ and $\widetilde{H}_2^\alpha(D)$ if $0 \leq \alpha \leq 1$, one may for instance follow the arguments of [81], Theorem 4.9.2. Using the duality relation $(\widetilde{H}_2^\alpha(D))' = H_2^{-\alpha}(D)$, on may conclude that for $0 < \kappa < 1$ and

$$-3/2 < \alpha \leq \alpha + \kappa < 3/2 \ ,$$

the fractional power $A^{\kappa/2}$, of the operator A maps $\overline{H}_2^{\alpha+\kappa}(D, \mathbb{C}^k)$ isomorphically onto $\overline{H}_2^\alpha(D, \mathbb{C}^k)$, leading to a scale of Banach spaces, see [81].

For $-1/2 < \alpha < 3/2$, the norms $\|\cdot\|_\alpha$ and $f \mapsto \left\|A^{\alpha/2} f\right\|_0$ are equivalent, for $-3/2 < \alpha \leq -1/2$, $\left\|\cdot|H_2^\alpha(D, \mathbb{C}^k)\right\|$ and $f \mapsto \left\|A^{\alpha/2} f\right\|_0$ are equivalent.

If $(P(t))_{t\geq 0}$ is the analytic semigroup of negative type on $L_2(D, \mathbb{C}^k)$ generated by $-A$, these isomorphism properties together with (2.10) permit to consider $(P(t))_{t\geq 0}$ as a strongly continuous and equibounded semigroup on $\overline{H}_2^\alpha(D, \mathbb{C}^k)$ for any fixed $-3/2 < \alpha < 3/2$.

2.3.5 Pointwise multiplication

In general the product of two arbitrary distributions does not make sense. However, in the special case of the spaces we use, one can define products via paraproducts, see [75] or [79].
Choose a function $\psi \in \mathcal{S}(\mathbb{R}^n)$ with $0 \leq \psi(x) \leq 1$ and such that $\psi(x) = 1$ if $|x|_n \leq 1$ and $\psi(x) = 0$ if $|x|_n \geq 3/2$. Given $f \in \mathcal{S}'(\mathbb{R}^n)$, consider

$$S^j f(x) := \left(\psi(2^{-j}\xi) f^\wedge\right)^\vee (x) \ ,$$

which, according to the Paley-Wiener-Schwartz theorem, is an entire analytic function for any $j \in \mathbb{N}$. The product fg of $f, g \in \mathcal{S}'(\mathbb{R}^n)$ is defined as

$$fg := \lim_{j \to \infty} S^j f S^j g \ ,$$

whenever the limit exists in $\mathcal{S}'(\mathbb{R}^n)$. The appropriate convergence is part of statements such as the lemma below. We refer to [75], Chapter 4, and use a special case of their Theorem 4.4.3/1. To indicate how some results of the present thesis can be generalized to an L_p-setting (in space), we state it for arbitrary $1 < p < \infty$:

Lemma 2.3.1. *Let $1 < p, q < \infty$, $0 < \beta < \delta$. Assume further that $q > p \vee (n/\delta)$. Then we have*

$$\left\| fg | H_p^{-\beta}(\mathbb{R}^n) \right\| \leq c \left\| f | H_p^{\delta}(\mathbb{R}^n) \right\| \left\| g | H_q^{-\beta}(\mathbb{R}^n) \right\|$$

for $f \in H_p^\delta(\mathbb{R}^n)$ and $g \in H_q^{-\beta}(\mathbb{R}^n)$.

For example, suppose that h is a compactly supported $(1-\beta')$-Hölder continuous function on $\mathbb{R}^n$, $0 < \beta' < 1$. By (2.20) it is seen to be a member of $H_q^{1-\beta}(\mathbb{R}^n)$ *for any* $1 < q < \infty$, provided $\beta' < \beta$. If so, it has partial derivatives $\frac{\partial h}{\partial x_l} \in H_q^{-\beta}(\mathbb{R}^n)$ which may be considered in place of g. By the freedom of choice for q, the statement of the Lemma can then be obtained for any $1 < p < \infty$.

The pointwise product preserves locality in the following sense, see [75], Lemma 4.2:

Lemma 2.3.2. *If $f, g \in \mathcal{S}'(\mathbb{R}^n)$ and supp $f \in \overline{D}$, then also supp $fg \in \overline{D}$.*

For $f, g \in \mathcal{S}'(\mathbb{R}^n, \mathbb{C}^k)$, $f = (f^1, ..., f^k)$, $g = (g^1, ..., g^k)$, we define the product $f \cdot g$ in the sense of (2.2),

$$f \cdot g := (f^1 g^1, ..., f^k g^k) \ .$$

The quoted results carry over.
Now consider

$$H^\alpha_{p,\infty}(\mathbb{R}^n, \mathbb{C}^k) := H^\alpha_p(\mathbb{R}^n, \mathbb{C}^k) \cap L_\infty(\mathbb{R}^n, \mathbb{C}^k)$$

and

$$\mathring{H}^\alpha_{p,\infty}(D, \mathbb{C}^k) := \mathring{H}^\alpha_p(D, \mathbb{C}^k) \cap L_\infty(\mathbb{R}^n, \mathbb{C}^k) \ .$$

In the case of $p = 2$, the space $H^\alpha_{2,\infty}(\mathbb{R}^n, \mathbb{C}^k)$ is endowed with the norm $\|\cdot\|_{\alpha,\infty} := \|\cdot\|_\alpha + \|\cdot\|_\infty$.

With the entry-wise product (2.2), $H^\alpha_{p,\infty}(\mathbb{R}^n, \mathbb{C}^k)$, $\alpha > 0$, is a *multiplication algebra*. For $p = 2$ that means in particular that

$$\|w \cdot v\|_\alpha \leq c \, \|w\|_{\alpha,\infty} \, \|v\|_{\alpha,\infty}$$

for any $v, w \in H^\alpha_\infty(\mathbb{R}^n, \mathbb{C}^k)$. See [75], 4.6.4/2 for the case $k = 1$.

2.3.6 Real subspaces and composition operators

We follow again [75]. For $z = x + iy \in \mathbb{C}$, $\overline{z} = x - iy$ denotes the complex conjugate of z, and we use similar notation for vectors or functions.
Given $f \in \mathcal{S}'(\mathbb{R}^n, \mathbb{C}^k)$, the distribution $\overline{f}$ is defined by requiring $\overline{f}(\varphi) = \overline{f(\overline{\varphi})}$ for any $\varphi \in \mathcal{S}(\mathbb{R}^n, \mathbb{C}^k)$. The *space of* $\mathbb{R}^k$*-valued Schwartz distributions* $\mathcal{S}'(\mathbb{R}^n, \mathbb{R}^k)$ is defined by

$$\mathcal{S}'(\mathbb{R}^n, \mathbb{R}^k) := \left\{ f \in \mathcal{S}'(\mathbb{R}^n, \mathbb{C}^k) : \overline{f} = f \right\} \ .$$

For $1 < p < \infty$, $\alpha \in \mathbb{R}$, set

$$H^\alpha_p(\mathbb{R}^n, \mathbb{R}^k) := H^\alpha_p(\mathbb{R}^n, \mathbb{C}^k) \cap \mathcal{S}'(\mathbb{R}^n, \mathbb{R}^k) \ .$$

This is a closed subspace of $H^\alpha_p(\mathbb{R}^n, \mathbb{C}^k)$. If $\alpha \geq 0$, i.e. if $f \in H^\alpha_p(\mathbb{R}^n, \mathbb{C}^k)$ may be seen as locally integrable function, we have $f \in H^\alpha_p(\mathbb{R}^n, \mathbb{R}^k)$ if and only if f is an $\mathbb{R}^k$-valued function in the ordinary sense.

In the cases we consider, approximation by smooth functions immediately shows that the product $f \cdot g$ is an $\mathbb{R}^k$-valued distribution, provided f and g are.

Given a function $G : \mathbb{R}^k \to \mathbb{R}$ with $G(0) = 0$ and having bounded differential $DG \in L_\infty(\mathbb{R}^k, \mathbb{R}^k)$, we define the *composition operator* $T_G : H_p^\alpha(\mathbb{R}^n, \mathbb{R}^k) \to H_p^\alpha(\mathbb{R}^n, \mathbb{R}^k)$, $1 < p < \infty$, $0 < \alpha < 1$, by

$$T_G f := G(f) = G(f_1, ..., f_k) \ .$$

The written mapping property is guaranteed by the mean value theorem and (2.20).

2.4 Fractional Brownian sheets

We survey some simple Gaussian random fields that serve as prototypes for the driving fields in our equations. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. See [6], [40], [48], [53] or [88] for information on the following.

2.4.1 Fractional Brownian fields

An $\mathbb{R}$-valued *fractional Brownian field* $B^\alpha = \{B^\alpha(x) : x \in \mathbb{R}^n\}$ of order $0 < \alpha < 1$ on $\mathbb{R}^n$ is a random field $B^\alpha : \Omega \times \mathbb{R}^n \to \mathbb{R}$ such that B^α is Gaussian with mean zero, and the covariance function is given by

$$\mathbb{E}\left\{B^\alpha(x)B^\alpha(y)\right\} = \frac{1}{2}\left(|x|_n^{2\alpha} + |y|_n^{2\alpha} - |y-x|_n^{2\alpha}\right) \ , \tag{2.24}$$

$x, y \in \mathbb{R}^n$. Recall that $|\cdot|_n$ denotes the Euclidean norm on $\mathbb{R}^n$. (2.24) in particular implies that a.s. $B^\alpha(0) = 0$. We refer to [53]. For $\alpha = 1/2$ we obtain Lévy's n-parameter Brownian motion, for $n = 1$ (and $0 < \alpha < 1$) the fractional Brownian motion, see [47], [55]. The restriction to a fixed line in $\mathbb{R}^n$ also yields a fractional Brownian motion, up to a constant.

2.4.2 Anisotropic fractional Brownian sheets

An $\mathbb{R}$-valued *anisotropic fractional Brownian sheet* $B^{\overline{\beta}} = \left\{B^{\overline{\beta}}(x) : x \in \mathbb{R}^n\right\}$ on $\mathbb{R}^n$ of order $\overline{\beta} = (\beta_1, ..., \beta_n)$, $0 < \beta_l < 1$, $l = 1, ..., n$, is a random field

$B^{\overline{\beta}} : \Omega \times \mathbb{R}^n \to \mathbb{R}$ such that $B^{\overline{\beta}}$ is Gaussian with mean zero, and the covariance function is given by

$$\mathbb{E}\left\{B^{\overline{\beta}}(x)B^{\overline{\beta}}(y)\right\} = \prod_{l=1}^{n} \frac{1}{2}\left(|x_l|^{2\beta_l} + |y_l|^{2\beta_l} - |y_l - x_l|^{2\beta_l}\right) ,$$

$x, y \in \mathbb{R}^n$.
For later purposes it is convenient to consider also $\mathbb{R}$-valued *anisotropic fractional Brownian sheets* $B^{\alpha,\overline{\beta}} = \left\{B^{\alpha,\overline{\beta}}(t,x) : (t,x) \in \mathbb{R}^{n+1}\right\}$ on $\mathbb{R}^{n+1}$ of orders $0 < \alpha < 1$ and $\overline{\beta} = (\beta_1, ..., \beta_n)$, $0 < \beta_l < 1$, $l = 1, ..., n$, the random fields $B^{\alpha,\overline{\beta}} : \Omega \times \mathbb{R}^{n+1} \to \mathbb{R}$ such that $B^{\alpha,\overline{\beta}}$ is Gaussian with mean zero and covariance function

$$\begin{aligned}\mathbb{E}&\left\{B^{\alpha,\overline{\beta}}(t,x)B^{\alpha,\overline{\beta}}(s,y)\right\} \\ &= \frac{1}{2}\left(|t|^{2\alpha} + |s|^{2\alpha} - |t-s|^{2\alpha}\right) \prod_{l=1}^{n} \frac{1}{2}\left(|x_l|^{2\beta_l} + |y_l|^{2\beta_l} - |y_l - x_l|^{2\beta_l}\right) ,\end{aligned}$$

$s, t \in \mathbb{R}$, $x, y \in \mathbb{R}^n$. A.s. $B^{\overline{\beta}}(0) = 0$ and $B^{\alpha,\overline{\beta}}(0) = 0$.
For $\alpha = \beta_1 = ... = \beta_n = 1/2$, one obtains the *Brownian sheet*.

2.4.3 Hybrid fractional Brownian sheets

A random field $B^{\alpha,\beta} = \left\{B^{\alpha,\beta}(t,x) : (t,x) \in \mathbb{R}^{n+1}\right\}$, $B^{\alpha,\beta} : \Omega \times \mathbb{R}^{n+1} \to \mathbb{R}$ with $0 < \alpha, \beta < 1$, such that $B^{\alpha,\beta}$ is Gaussian with mean zero and the covariance function is given by

$$\begin{aligned}\mathbb{E}&\left\{B^{\alpha,\beta}(t,x)B^{\alpha,\beta}(s,y)\right\} \\ &= \frac{1}{2}\left(|t|^{2\alpha} + |s|^{2\alpha} - |t-s|^{2\alpha}\right) \frac{1}{2}\left(|x|_n^{2\beta} + |y|_n^{2\beta} - |y-x|_n^{2\beta}\right) ,\end{aligned} \tag{2.25}$$

$s, t \in \mathbb{R}$, $x, y \in \mathbb{R}^n$, will be called an $\mathbb{R}$-valued *hybrid fractional Brownian sheet* of orders $0 < \alpha, \beta < 1$ on $\mathbb{R}^{n+1}$.

Recall that $|\cdot|_n$ is the Euclidean norm on $\mathbb{R}^n$ and $|\cdot|$ the absolute value on $\mathbb{R}$. In particular, $B^{\alpha,\beta}(0) = 0$ a.s. For the case $n = 1$ we obtain an anisotropic fractional Brownian sheet on $\mathbb{R}^2$.

2.4.4 Hölder continuity

For $(t,x) \in (a,b) \times \mathbb{R}^n$, $u \in \mathbb{R}$ small enough in modulus and $r \in \mathbb{R}^n$, let

$$\Delta_{u,r} g(t,x) := g(t+u, x+r) - g(t+u, x) - g(t, x+r) + g(t,x) \tag{2.26}$$

denote the 'rectangular' increments of a real-valued function g on $(a,b) \times \mathbb{R}^n$. $|\cdot|_n$ denotes the Euclidean norm in $\mathbb{R}^n$, n is suppressed from notation if $n = 1$.

Now let $U \subset \mathbb{R}$ be an open neighbourhood of $[a,b]$, let $g : U \times \mathbb{R}^n \to \mathbb{R}$ and assume there is a compact set $K \subset \mathbb{R}^n$ such that for any $t \in U$, the support of g is contained in K. Let $0 < \alpha' < 1$, $0 < \beta' < 1$, $0 < \beta'_l < 1$, and consider the *multiple Hölder conditions*

$$|\Delta_{u,r} g(t,x)| \le c|u|^{\alpha'} |r|_n^{\beta'} \ , \tag{2.27}$$

and

$$|\Delta_{u,se_l} g(t,x)| \le c|u|^{\alpha'} |s|^{\beta'_l} \ , \quad l = 1, ..., n \ , \tag{2.28}$$

as well as the *simple Hölder conditions*

$$|g(t+u, x) - g(t,x)| \le c|u|^{\alpha'} \quad , \tag{2.29}$$

$$|g(t, x+r) - g(t,x)| \le c|r|_n^{\beta'} \quad , \tag{2.30}$$

and

$$|g(t, x+se_l) - g(t,x)| \le c|s|^{\beta'_l} \quad , \quad l = 1, ..., n \ , \tag{2.31}$$

for all $(t,x) \in [a,b] \times \mathbb{R}^n$, small $s, u \in \mathbb{R}$, $r \in \mathbb{R}^n$ and with a universal constant $c > 0$ depending only on $[a,b]$ and K.

Lemma 2.4.1. *Let $K \subset \mathbb{R}^n$ be an arbitrary compact set and $[a,b] \subset \mathbb{R}$.*

(i) The fractional Brownian field B^α on $\mathbb{R}^n$ has a modification, again denoted by B^α, whose paths are α'-Hölder continuous on K a.s. for any $0 < \alpha' < \alpha$.

(ii) The anisotropic sheet $B^{\alpha,\overline{\beta}}$ on $\mathbb{R}^{n+1}$, $\overline{\beta} = (\beta_1, ..., \beta_n)$, possesses a modification, again denoted by $B^{\alpha,\overline{\beta}}$, whose paths on $[a,b] \times K$ a.s. fulfill the Hölder conditions (2.28), (2.29) and (2.31) in place of g for all $0 < \alpha' < \alpha$ and $0 < \beta'_l < \beta_l$. An analogous assertion is true for the anisotropic sheet $B^{\overline{\beta}}$ on $\mathbb{R}^n$.

(iii) The hybrid sheet $B^{\alpha,\beta}$ on $\mathbb{R}^{n+1}$ possesses a modification, again denoted by $B^{\alpha,\beta}$, whose paths on $[a,b]\times K$ a.s. fulfill the Hölder conditions (2.27), (2.29) and (2.30) in place of f_l for all $0<\alpha'<\alpha$ and $0<\beta'<\beta$.

The constants $c>0$ in (2.27)-(2.31) possibly depend on ω.

(i) follows from (2.24) by Kolmogorov-Chentsov and the Gaussian property. Multiparameter variants of these arguments yield (ii) and (iii), see for instance [6], [29] or [52].

Remark 2.4.1. Given a bounded domain D in $\mathbb{R}^n$, choose K such that it contains an open neighbourhood of $\overline{D}$. Let $\varphi\in C_0^\infty(\mathbb{R}^n)$ be such that it is supported in K, $0\le\varphi(x)\le 1$, $x\in\mathbb{R}^n$ and $\varphi(x)=1$ for $x\in\overline{D}$. Obviously the pointwise products $\varphi B^\alpha(\omega)$, $\varphi B^{\alpha,\beta}(\omega)$ and $\varphi B^{\alpha,\overline{\beta}}(\omega)$ still satisfy the mentioned Hölder conditions for $\mathbb{P}$-a.e. $\omega\in\Omega$. For our purposes, it is no loss to correct the paths by such a simple *cut-off*. We assume this has been done and (abusing notation) write again $B^\alpha(\omega)$, $B^{\alpha,\beta}(\omega)$ and $B^{\alpha,\overline{\beta}}(\omega)$, respectively.
This cut-off permits to neglect the 'bad' asymtotic behaviour of fractional Brownian sheets at infinity, cf. [7].

Chapter 3

Stieltjes type integrals

In [91], [92] and [66] pathwise integrals had been introduced by means of fractional calculus, they had been used to study related SDE's. Basically the same construction yields an effective tool also for the study of parabolic SPDE's, provided it is formulated in a suitable vector-valued sense.

3.1 Integrals via fractional calculus

Again, let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be two separable Banach spaces and $L = L(E, F)$ the space of bounded linear operators with the usual operator norm $\|\cdot\|_L$.

3.1.1 Forward integrals

We start with a Stieltjes integral as introduced [91], now for vector-valued functions. For $f : (a, b) \to E$ we suppose $f(a+) = \lim_{\delta \to 0} f(a + \delta)$ exists in the strong sense and put $f_{a+} = \mathbf{1}_{(a,b)}(t)(f(t) - f(a+))$. For $U : (a, b) \to L$ we assume that $U(a+)f = \lim_{\delta \to 0} U(a + \delta)f$ for any $f \in E$ exists as the limit in the strong sense in F, then consequently $U(a+) \in L$, too. Set $U_{a+}(t) = \mathbf{1}_{(a,b)}(t)(U(t) - U(a+))$. The meanings of f_{b-} and U_{b-} are similar.

Definition 3.1.1. Suppose $0 \le \alpha \le 1$, $p, q \ge 1$, $1/p + 1/q \le 1$. Let $f : (a, b) \to E$ be such that $f_{a+} \in I^{\alpha}_{a+}(L_p((a, b), E))$ and $U : (a, b) \to L$ such that $U_{b-} \in I^{1-\alpha}_{b-}(L_q((a, b), L))$. Define the *forward integral* of the E-valued function f with respect to the operator-valued function U by

$$\int_a^b f(t)dU(t) := (-1)^\alpha \int_a^b D_{b-}^{1-\alpha} U_{b-}(t)\, D_{a+}^\alpha f_{a+}(t)\, dt + U(b-)f(a+) - U(a+)f(a+) \quad (3.1)$$

The integral is directed forward.

Remark 3.1.1. (i) As in [91], (2.7) can be used to show that the definition (3.1) is correct, i.e. does not depend of the particular choice of $0 \leq \alpha \leq 1$.

(ii) To justify the notation of the marginal limits in (3.1), apply the triangle inequality to $\|U(a+\varepsilon)f(a+\delta) - U(a+)f(a+)\|_F$ for $\varepsilon > 0$ and $\delta > 0$ and consider

$$\|(U(a+\varepsilon) - U(a+))f(a+)\|_F$$

as well as

$$\|U(a+\varepsilon)(f(a+\delta) - f(a+))\|_F .$$

For the first term obviously the order of the limit processes is arbitrary, for the second term this follows from the existence of $U(a+) \in L$ in the strong sense and since $U(t) \in L$, $t \in (a,b)$.

(iii) In the case that $0 \leq \alpha < 1/p$, the entire right hand side in (3.1) equals the integral with just f in place of f_{a+} (then without correction terms).

(iv) If real Banach spaces are considered, the integrals given by (3.1) are real-valued, due to the definition of the fractional derivatives.

Let E' denote the dual space of E. For $g \in E'$, let $\langle f, g\rangle$ denote the *dual pairing* of $f \in E$ and $g \in E'$. Given a function $g : (a,b) \to E'$,

$$U(t) := \langle \cdot, g(t)\rangle \ , \ t \in (a,b) \ , \quad (3.2)$$

defines a bounded operator-valued function $U : (a,b) \to E$. We assume there is some $g(a+) \in E'$ such that $\langle f, g(a+)\rangle = \lim_{\delta\to 0} \langle f, g(a+\delta)\rangle$ for any $f \in E$ and put $g_{a+}(t) = \mathbf{1}_{(a,b)}(t)(g(t) - g(a+))$. Specifying Definition 3.1.1 to this case with $g : (a, b) \to E'$, $g_{b-} \in I_{b-}^{1-\alpha}(L_q((a, b), E'))$ and α, p, q, f as before, the forward integral according to (3.1) equals

$$\int_a^b \langle f(t), dg(t)\rangle := (-1)^\alpha \int_a^b \left\langle D_{a+}^\alpha f_{a+}(t), D_{b-}^{1-\alpha} g_{b-}(t)\right\rangle dt + \langle f(a+), g(b-)\rangle - \langle f(a+), g(a+)\rangle \ . \tag{3.3}$$

For $E = E' = \mathbb{R}$ we arrive at the integral considered in [91] and [92].

3.1.2 Average integrals

We state two limit representations and define an average version.

Lemma 3.1.1. *Let* $1/p + 1/q \leq 1$.
(i) Under the assumptions of Definitions 3.1.1 we have

$$\int_a^b f(t)dU(t) = \lim_{\varepsilon \to 0} \int_a^b I_{a+}^\varepsilon f(t) dU(t) \ .$$

(ii) Suppose $0 < \alpha < 1$, $0 < \varepsilon < \alpha$, $f \in I_{a+}^{\alpha-\varepsilon}(L_p((a, b), E))$, $\alpha p \neq 1$ *and* $U_{b-} \in I_{b-}^{1-\alpha}(L_q((a, b), L))$. *Then we have*

$$\int_a^b I_{a+}^\varepsilon f(t)dU(t) = \frac{(1-\varepsilon)}{\Gamma(\varepsilon)} \int_0^\infty t^{\varepsilon-1} \int_a^b (U_{b-}(s+t) - U_{b-}(s)) f(s)\, ds\, \frac{dt}{t} \ ,$$

the integral $\int_0^\infty$ *taken in the sense of principal values.*

The proof of the Lemma resembles that of the scalar valued case, we refer to [92], Lemma 4.1 and 4.2.
In view of the asymptotics of the Gamma function ε may replace $1/\Gamma(\varepsilon)$ when considering the limit as $\varepsilon \to 0$. We set

$$(A) \int_a^b f(t)dU(t) := \lim_{\varepsilon\to 0} \varepsilon \int_0^1 t^{\varepsilon-1} \int_a^b (U_{b-}(s+t) - U_{b-}(s)) f(s)\, ds \frac{dt}{t} \tag{3.4}$$

for measurable functions $f : (a, b) \to E$ and $U : (a, b) \to L$ whenever the right hand side is well defined and exists. This is an extension of Definition 3.1.1. The notation (A) means 'average'.

3.1.3 Riemann-Stieltjes integrals in Banach spaces

The integrals defined in (3.1) extend Riemann-Stieltjes integrals in Banach spaces as studied in [33]. There the following construction had been introduced:
Let E and F be real Banach spaces normed by $\|\cdot\|_E$ and $\|\cdot\|_F$, respectively, and $L(E,F)$ the space of bounded linear operators from E into F, endowed with the operator norm.

Let $(a,b) \subset \mathbb{R}$ be a bounded interval and consider two functions $f : (a,b) \to E$ and $U : (a,b) \to L(E,F)$. We assume throughout that the limits $f(a+)$,$f(b-)$, $U(a+)$ and $U(b-)$ exist in the respective spaces and set $f(a) := f(a+)$ etc. below.
Let $\mathcal{P}_\Delta = \{t_i : i = 0, ..., k,\ a = t_0 < t_1 < ... < t_k = b\}$ with some $k \in \mathbb{N}$ be a partition of (a,b) with $\max_i |t_i - t_{i-1}| < \Delta$. If the Riemann-Stieltjes sums

$$S(f,U,\mathcal{P}_\Delta) = \sum_{i=1}^{k} [(U(t_i) - U(t_{i-1}))f(\tau_i)] \ ,$$

where $\tau_i \in [t_{i-1}, t_i]$, converge to a limit along a sequence of refining partitions $\mathcal{P}_\Delta$ of the above type as Δ goes to zero, this limit is called *Riemann-Stieltjes integral (in the sense of Gowurin)* and denoted by

$$(RS) \int_a^b f dU := \lim_{\Delta \to 0} S(f,U,\mathcal{P}_\Delta) \ .$$

A function $U : (a,b) \to L(E,F)$ is said to have the *ω-property on* (a,b), if there exists some $M > 0$ such that for any partition $\mathcal{P}$ of the above type and any $x_i \in E$, $i = 0, ..., k-1$,

$$\left\| \sum_{i=1}^{k} (U(t_i) - U(t_{i-1}))x_i \right\|_F \leq M \ \max_i \|x_i\|_E \ .$$

There are simple existence conditions similar to the sclalar-valued case. One can show that $(RS)\int_a^b f dU$ exists if f is strongly continuous and U has the ω-property. It also exists if U is strongly continuous and f is of *bounded variation on* (a,b), i.e. there exists some $M > 0$ such that

$$\sup_{\mathcal{P}} \sum_{i=1}^{k} \|f(t_i) - f(t_{i-1})\|_E \leq M \ ,$$

the supremum taken over all partitions of (a, b).
The concept extends to complex Banach spaces E and F in the coordinate-wise sense.

The following simple lemma holds:

Lemma 3.1.2. *Suppose $U : (a, b) \to L(E, F)$ is uniformly bounded on (a, b) and has the ω-property. Assume $\Phi : (a, b) \times (a, b) \to E$ is a function such that for a.e. $\tau \in (a, b)$, $\Phi(\cdot, \tau)$ is strongly continuous on $(a, b) \setminus \{\tau\}$ and*

$$\sup_{t\in(a,b)} \int_a^b \|\Phi(t, \tau)\|_E \, d\tau < \infty \; . \tag{3.5}$$

Then both integrals below exist and

$$(RS) \int_a^b \int_a^t \Phi(t, \tau) d\tau dU(t) = \int_a^b (RS) \int_\tau^b \Phi(t, \tau) dU(t) d\tau \; .$$

Proof. By (3.5), $\Psi(t) = \int_a^t \Phi(t, \tau)d\tau$ is strongly continuous on (a, b) and with $\tau_i \in [t_{i-1}, t_i]$, the right hand side rewrites

$$\begin{aligned} \int_a^b \Psi(t)dU(t) &= \lim_{\Delta\to 0} \sum_{i=1}^k \int_a^{\tau_i} (U(t_i) - U(t_{i-1}))\Phi(\tau_i, \tau) \, d\tau \\ &= \lim_{\Delta\to 0} \int_a^b \sum_{i=1}^k (U(t_i) - U(t_{i-1}))\mathbf{1}_{[\tau+\varepsilon,b)}(\tau_i)\Phi(\tau_i, \tau) \, d\tau \\ &\quad + \lim_{\Delta\to 0} \int_a^b \sum_{i=1}^k (U(t_i) - U(t_{i-1}))\mathbf{1}_{(\tau,\tau+\varepsilon)}(\tau_i)\Phi(\tau_i, \tau) \, d\tau \end{aligned} \tag{3.6}$$

for any $\varepsilon > 0$. We have use the uniform boundedness of U in the first equality. By the ω-property of U, for any $a \le c < d \le b$,

$$\left\| \sum_{i=1}^k (U(t_i) - U(t_{i-1}))\mathbf{1}_{(c,d)}(\tau_i)\Phi(\tau_i, \tau) \right\|_F \le M \, \max_i \mathbf{1}_{(c,d)}(\tau_i) \, \|\Phi(\tau_i, \tau)\|_F$$

and by (3.5),

$$\begin{aligned} &\int_a^b \left\| \sum_{i=1}^k (U(t_i) - U(t_{i-1}))\mathbf{1}_{(c,d)}(\tau_i)\Phi(\tau_i, \tau) \right\|_F \, d\tau \\ &\qquad\qquad \le M \, \max_i \int_a^b \mathbf{1}_{(c,d)}(\tau_i) \, \|\Phi(\tau_i, \tau)\|_E \, d\tau < \infty \; . \end{aligned}$$

Therefore the first summand in (3.6) is $\int_a^b \int_{\tau+\varepsilon}^b \Phi(t,\tau)dU(t)\, d\tau$, while the second is bounded by $M \sup_{t\in(a,b)} \int_{(t-\varepsilon)\vee a}^t \|\Phi(t,\tau)\|_E \, d\tau$, which tends to zero as ε does. □

With the aid of this lemma it can be seen that for suitable functions f and U the integral according to Definition 3.1.1 coincides with the Riemann-Stieltjes integral. Using Lemma 3.1.2 one can follow the lines of [91], Theorem 2.4.(i) to obtain:

Proposition 3.1.1. *Suppose $U : (a,b) \to L(E,F)$ has the ω-property and $f : (a,b) \to E$ is strongly continuous. Assume further, f and U satisfy the hypotheses of Definition 3.1.1 and with some $0 \le \alpha \le 1$,*

$$\sup_{t\in(a,b)} \int_a^b \left\| D_{a+}^{\alpha} f_{a+}(\tau) \right\|_E (t-\tau)^{\alpha-1} d\tau < \infty \ .$$

Then $\int_a^b f dU = (RS) \int_a^b f dU$.

3.1.4 Some remarks

It is worth pointing out that a slight modification of Definition 3.1.1 is closely related to the integral operator we will construct later on. Let us swap the roles of the functions in that definition and consider slightly different hypotheses: Assume $g : (a,b) \to E$ and $U : (a,b) \to L$ are such that $g(a+)$, $g(b-)$ and $U(a+)$ exist (in the respective sense discussed there). Suppose $0 \le \alpha \le 1$, $g_{b-} \in I_{b-}^{\alpha}(L_\infty((a,b),E)))$ and $U_{a+} \in I_{a+}^{1-\alpha}(L_1((a,b),L)))$. Then we can define the *forward integral* of the L-valued function U with respect to the E-valued function g by

$$\int_a^b U(t)dg(t) := (-1)^{\alpha} \int_a^b D_{a+}U_{a+}(t) D_{b-}^{1-\alpha} g_{b-}(t)dt + U(a+)(g(b-) - g(a+)) \ . \tag{3.7}$$

Remark 3.1.2. (i) As before, the hypotheses imply the existence of the integral.

(ii) Variants of (3.7) will be used to obtain mild solutions to parabolic problems: With $a = 0$ and $b = t > 0$ we will consider the time reversed semigroup $U(s) = P(t-s)$ (in the case of additive noise) or the composition $U(s) = P(t-s) \circ M_{h(s)}$ of the time-reversed semigroup with some

multiplication operator $M_{h(s)}z := h(s)z$ (in the case of multiplicative noise).
This idea becomes meaningful if one chooses suitable Sobolev spaces, a good product definition and performs some additional spatial differentiation of $D_{b-}^{1-\alpha}g_{b-}(t)$ (in distributional sense). This will be discussed in Chapter 4.

(iii) We prefer to prove the correctness of definition (3.7) in the particular applications.

3.2 Sobolev spaces and duality

This section introduces forward integrals of $\mathbb{R}^n$-valued fields f w.r.t. $\mathbb{R}$-valued functions g over smooth bounded domains $D \subset \mathbb{R}^n$. The special case $n = 1$ yields the forward integral as familiar from [76].
The forward integral is defined and using simple Fourier multiplier arguments, conditions sufficient for its existence are proved. Limit representations are shown, which lead to an average version.

3.2.1 Forward integrals

Let $D \subset \mathbb{R}^n$ be a bounded domain. Let $f = (f_1, ..., f_n)$ be an $\mathbb{R}^n$-valued vector field on $\mathbb{R}^n$ and g an $\mathbb{R}$-valued function on $\mathbb{R}^n$. For fixed $l = 1, ..., n$ denote the *'forward differences'* of g in direction e_l by

$$\partial_{l,r}^{+} g(x) := \frac{1}{r}\left(g(x + re_l) - g(x)\right) \ , \ r > 0 \ . \tag{3.8}$$

Define the *'forward pre-gradient'* $\nabla_r^+ g$, $r > 0$, of the function g by

$$\nabla_r^+ g(x) := \left(\partial_{1,r}^+ g(x), ..., \partial_{n,r}^+ g(x)\right) \ , \ r > 0 \ . \tag{3.9}$$

Writing $\langle \cdot, \cdot \rangle$ for the standard scalar product, we have $\langle f(x), \nabla_r^+ g(x) \rangle = \sum_{l=1}^n f_l(x), \partial_{l,r}^+ g(x)$ for $f = (f_1, ..., f_n)$.

Definition 3.2.1. Given an $\mathbb{R}^n$-valued function $f = (f_1, ..., f_n)$ on $\mathbb{R}^n$ and an $\mathbb{R}$-valued function g on $\mathbb{R}^n$, the *(partial) forward integral of f w.r.t. to g on D* is defined as the limit

$$\int_D \left\langle f(x), \nabla^+ g(x) \right\rangle dx := \lim_{r \to 0} \int_D \left\langle f(x), \nabla_r^+ g(x) \right\rangle dx \ , \tag{3.10}$$

whenever it exists. We also use $\int_D \langle f, \nabla^+ g\rangle \, dx$ to denote this integral.

Examples 3.2.1. Suppose $n = 1$, $D = (a, b)$, f and g both are defined on $\mathbb{R}$, g continuous at b. Let

$$\int_a^b f d^- g := \lim_{r\to 0} \int_a^b f(x) \frac{g_b(x+r) - g_b(x)}{r} \, dx$$

with $g_b := \mathbf{1}_{(a,b)}(x)\,(g(x) - g(b))$ denote the forward integral of f w.r.t. g as introduced in [76], whenever the limit exists. Then

$$\int_a^b f d^- g = \int_{(a,b)} \left\langle f, \nabla^+ g\right\rangle dx \ .$$

We prefer +, indicating the right-sided derivative instead of the traditional '-' referring to the integration process.

3.2.2 Existence conditions

As before, let $f = (f_1, ..., f_n)$ be a $\mathbb{R}^n$-valued function on $\mathbb{R}^n$ and g a $\mathbb{R}$-valued function on $\mathbb{R}^n$. Usually f and g will not be differentiable, but members of some function spaces. Denoting by $\mathbf{1}_D$ the indicator function of D, we first record the following:

Lemma 3.2.1. *Let $D \subset \mathbb{R}^n$ be a bounded C^∞-domain and $h \in H_p^\alpha(\mathbb{R}^n)$ with $1 < p < \infty$, $0 < \alpha < 1/p$. Then*

$$\left\|\mathbf{1}_D h | H_p^\alpha(\mathbb{R}^n)\right\| \le c \left\|h | H_p^\alpha(\mathbb{R}^n)\right\|$$

with a constant $c > 0$ independent of h.

This is proved in [82]. In our case is not necessary that D is C^∞, C^1 would be sufficient. Consider the *gradient* ∇g of g,

$$\nabla g := \left(\frac{\partial g}{\partial x_1}, ..., \frac{\partial g}{\partial x_n}\right) \ ,$$

the partials taken in distributional sense. If g is such that for some $\alpha_l > 0$, $\frac{\partial g}{\partial x_l} \in H_{p',l}^{-\alpha_l}(\mathbb{R}^n) \subset H_{p'}^{-\alpha_l}(\mathbb{R}^n)$, and if f is such that $f_l \in H_p^{\alpha_l}(\mathbb{R}^n)$ with

$0 < \alpha_l < 1/p$, $l = 1, ..., n$, then $\left\langle \mathbf{1}_D f_l, \frac{\partial g}{\partial x_l} \right\rangle$ may be seen as dual pairing according to (2.19). We write

$$\langle \mathbf{1}_D f, \nabla g \rangle := \sum_{l=1}^{n} \left\langle \mathbf{1}_D f_l, \frac{\partial g}{\partial x_l} \right\rangle \ . \tag{3.11}$$

By $\overline{1}$ we denote the n-vector $(1, ..., 1)$. In this situation we can slightly refine (2.19) to obtain:

Proposition 3.2.1. *Let $1 < p < \infty$, $1/p + 1/p' = 1$ and let $\overline{\alpha} = (\alpha_1, ..., \alpha_n)$ fulfill $0 < \alpha_l \leq 1/p$, $l = 1, ..., n$. Suppose $f = (f_1, ..., f_n)$ is such that $f_l \in H_p^{\alpha_l}(\mathbb{R}^n)$ and $g \in H_{p'}^{\overline{1}-\overline{\alpha}}(\mathbb{R}^n)$. Then the forward integral (3.10) exists and with notation (3.11),*

$$\int_D \left\langle f, \nabla^+ g \right\rangle dx = \langle \mathbf{1}_D f, \nabla g \rangle \ .$$

Moreover, the estimate

$$\left| \int_D \left\langle f, \nabla^+ g \right\rangle dx \right| \leq c \sum_{l=1}^{n} \left\| f_l | H_p^{\alpha_l}(\mathbb{R}^n) \right\| \left\| g | H_{p',l}^{1-\alpha_l}(\mathbb{R}^n) \right\|$$

holds.

Given $a \in \mathbb{R}$, the translation T_a is defined by $(f \circ T_a)(x) := f(x + a)$.

Proof. The estimate for the dual pairing is obvious, we need to verify existence and value of the forward integral. It suffices to consider a single summand in (3.11). We fix an integer $1 \leq l \leq n$ and write α for α_l. For $r > 0$ set

$$I_l(f, g, r) = \int_{\mathbb{R}^n} \mathbf{1}_D f_l \, \partial_{l,r}^+ g dx = \int_{\mathbb{R}^n} \mathbf{1}_D f_l(x) \frac{g \circ T_{re_l} - g}{r}(x) \, dx \ .$$

First assume $g \in \mathcal{S}(\mathbb{R}^n)$. Then

$$I_l(f, g, r) = \int_0^1 \int_{\mathbb{R}^n} \left(\mathbf{1}_D f_l \right)^\wedge (\xi) \, e^{irt\xi_l} i\xi_l \, \hat{g}(\xi) \, d\xi \, dt \ .$$

For fixed $t > 0$, the inner integral in the last line equals

$$\int_{\mathbb{R}^n} \left((1 + \xi_l^2)^{\alpha/2} \left(\mathbf{1}_D f_l \right)^\wedge \right)^\vee (x) \left(i\xi_l (1 + \xi_l^2)^{-\alpha/2} \ e^{irt\xi_l} \, \hat{g} \right)^\vee (x) \, dx$$

and by Hölder's inequality this is (componentwise) bounded above by

$$\begin{aligned}
&\left\| \left((1+\xi_l^2)^{\alpha/2}\ (\mathbf{1}_D f_l)^\wedge\right)^\vee |L_p(\mathbb{R}^n)\right\| \left\| \left(i\xi_l(1+\xi_l^2)^{-\alpha/2}\, e^{irt\xi_l}\ \hat{g}\right)^\vee |L_{p'}(\mathbb{R}^n)\right\| \\
\le c &\left\| \left((1+\xi^2)^{\alpha/2}\ (\mathbf{1}_D f_l)^\wedge\right)^\vee |L_p(\mathbb{R}^n)\right\| \left\| \left((1+\xi_l^2)^{(1-\alpha)/2}\ \hat{g}\right)^\vee \circ T_{rte_l}|L_{p'}(\mathbb{R}^n)\right\| ,
\end{aligned}$$

note that $(1+\xi_l^2)^{\alpha/2}(1+\xi^2)^{-\alpha/2}$ and $\xi_l(1+\xi_l^2)^{-1/2}$ are Fourier multipliers. Hence by Lemma 3.2.1 and translation invariance of the $L_{p'}(\mathbb{R}^n)$-norm,

$$|I_l(f,g,r)| \le c\ \left\| f_l|H_p^\alpha(\mathbb{R}^n)\right\| \left\| g|H_{p',l}^{1-\alpha}(\mathbb{R}^n)\right\| \ . \tag{3.12}$$

For fixed $0<r<r'$ we similarly obtain

$$\begin{aligned}
&|I_l(f,g,r)-I_l(f,g,r')| \le c\ \left\| f_l|H_p^\alpha(\mathbb{R}^n)\right\| \times \\
&\times \int_0^1 \left\| \left((1+\xi_l^2)^{(1-\alpha)/2}\hat{g}\right)^\vee \circ T_{tre_l} - \left((1+\xi_l^2)^{(1-\alpha)/2}\hat{g}\right)^\vee \circ T_{tr'e_l}|L_{p'}(\mathbb{R}^n)\right\| \, dt.
\end{aligned} \tag{3.13}$$

Approximating $g \in H_{p',l}^{1-\alpha}(\mathbb{R}^n)$ by functions $\varphi_j \in \mathcal{S}(\mathbb{R}^n)$ in $H_{p',l}^{1-\alpha}(\mathbb{R}^n)$, (3.12) and (3.13) carry over, notice that by Hölder

$$|I_l(f,g-\varphi_j,r,w)| \le \|f_l|L_p(\mathbb{R}^n)\| \left\| \frac{g\circ T_{re_l}-g}{r} - \frac{\varphi_j\circ T_{re_l}-\varphi_j}{r}|L_{p'}(\mathbb{R}^n)\right\|$$

and φ_j tends to g in $L_{p'}(\mathbb{R}^n)$.
Finally, the right-hand side of (3.13) tends to zero as r and r' do, since for any $u \in L_{p'}(\mathbb{R}^n)$ and $h \in \mathbb{R}$, $\lim_{r\to 0}\|u-u(\cdot+rh)|L_{p'}(\mathbb{R}^n)\| = 0$. Therefore (3.10) exists. On the other hand, for $f \in H_p^\alpha(\mathbb{R}^n)$ and $g \in \mathcal{S}(\mathbb{R}^n)$

$$\begin{aligned}
&\left|\left\langle \mathbf{1}_D f_l, \frac{\partial g}{\partial x_l}\right\rangle - I_l(f,g,r)\right| \le c\ \left\| f_l|H_p^\alpha(\mathbb{R}^n)\right\| \times \\
&\qquad \times \int_0^1 \left\| \left(i\xi_l(1+\xi_l^2)^{-\alpha/2}\hat{g}\right)^\vee - \left(i\xi_l(1+\xi_l^2)^{-\alpha/2}\hat{g}\right)^\vee \circ T_{tre_l}|L_{p'}(\mathbb{R}^n)\right\| \, dt
\end{aligned}$$

tends to zero as r does by similar arguments, and completion yields the desired equality. □

Remark 3.2.1. (i) Similarly, the existence of backward integrals can be deduced.

(ii) As in Definition (3.4), we may also consider an average limit corresponding to (3.10): Given two functions $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}$, set

$$(A) \int_D \langle f, \nabla^+ g \rangle \, dx := \lim_{\varepsilon \to 0} \varepsilon \int_0^1 r^{\varepsilon - 1} \int_D \langle f(x), \nabla_r^+ g(x) \rangle \, dx dr \ , \quad (3.14)$$

whenever the right hand side exists. (A) stands for 'average'. As the existence of (3.10) implies that of (3.14), the latter extends the first. For $n = 1$, $D = (a, b)$ and with f, g according to Examples 3.2.1, (3.14) reads

$$\lim_{\varepsilon \to 0} \varepsilon \int_0^1 r^{\varepsilon - 1} \int_a^b f(x) \frac{g(x + r) - g(x)}{r} \, dx \, dr \ .$$

Recall the definition of the average integral via fractional calculus, (3.4). Specifying the case adressed in (3.2) and (3.3) further to such $f, g : \mathbb{R} \to \mathbb{R}$, these special cases of (3.14) and (3.4) yield the same.

3.3 Two-parameter integrals

This section gives a simple illustration how the discussed concepts can be combined. We define a basic Stieltjes-type integral for functions of variables $(t, x) \in (a, b) \times D$, where (a, b) is a finite interval and D a bounded C^∞-domain in $\mathbb{R}^n$. We 'mix' the above constructions and use the approach via fractional calculus for the variable $t \in (a, b)$ and the approach via forward differences and duality for the variable $x \in D$. Though seeming peculiar at first sight, this construction suits later studies of partial differential equations. There t will denote the time and x the space parameter.
In the special case where D itself is an interval, Stieltjes type integrals can also be constructed using two-parameter fractional calculus. This can be carried through along the lines of [91], we refer to [58], [77], [83] and to [28] for an application to PDEs.
Below the two-parameter integral is defined and limit statements are deduced, the latter allow a representation of Itô type integrals later on.
Functions depending on t and x will repeatedly be seen as vector-valued functions of t. If not mentioned otherwise, fractional calculus always applies to the variable $t \in (a, b)$.

3.3.1 Basic definition

Let $f = (f_1, ..., f_n) : (a, b) \times \mathbb{R}^n \to \mathbb{R}^n$ and $g : (a, b) \times \mathbb{R}^n \to \mathbb{R}$ be given. D is a bounded C^∞-domain in $\mathbb{R}^n$. We formulate conditions in terms of some hybrid function spaces under which we introduce the integral. Assume that with some $1 < p < \infty$, $1/p + 1/p' = 1$ and $0 \le \alpha, \beta_l \le 1$, $\beta_l < 1/p$, $l = 1, ..., n$ the following holds:

(I) f_l and g possess the strong limits $f_l(a+)$, $g(a+)$ and $g(b-)$ in $H_p^{\beta_l}(\mathbb{R}^n)$ and $H_{p'}^{\overline{1}-\overline{\beta}}(\mathbb{R}^n)$, respectively.

(II) $f_{l,a+} \in I_{a+}^{\alpha}(L_p((a, b), H_p^{\beta_l}(\mathbb{R}^n))$ and $g_{b-} \in I_{b-}^{\alpha}(L_{p'}((a, b), H_{p'}^{\overline{1}-\overline{\beta}}(\mathbb{R}^n))$, where $f_{l,a+}(x) := \mathbf{1}_{(a,b)}(x)(f_l(x) - f_l(a+))$, and g_{b-} is defined similarly.

We recall the Hölder conditions (2.27)-(2.31) that had been used in connection with some random fields. For convenience we sketch them again, for the precise formulation see Section 2.4:

$$|\Delta_{u,r} g(t, x)| \le c|u|^{\alpha'} |r|_n^{\beta'} , \tag{3.15}$$

$$|\Delta_{u,se_l} g(t, x)| \le c|u|^{\alpha'} |s|^{\beta'_l} , \quad l = 1, ..., n , \tag{3.16}$$

$$|g(t + u, x) - g(t, x)| \le c|u|^{\alpha'} , \tag{3.17}$$

$$|g(t, x + r) - g(t, x)| \le c|r|_n^{\beta'} , \tag{3.18}$$

$$|g(t, x + se_l) - g(t, x)| \le c|s|^{\beta'_l} , \quad l = 1, ..., n . \tag{3.19}$$

By $\Delta_{u,r} g(t, x) := g(t+u, x+r) - g(t+u, x) - g(t, x+r) + g(t, x)$ we had denoted the rectangular increment of a function g of two parameters $t \in U \subset \mathbb{R}$ and $x \in \mathbb{R}^n$.

Lemma 3.3.1. *Let $U \subset \mathbb{R}$ be an open neighbourhood of $[a, b]$, $f = (f_1, ..., f_n) : U \times \mathbb{R}^n \to \mathbb{R}^n$ and $g : U \times \mathbb{R}^n \to \mathbb{R}$, and assume there is a compact set $K \subset \mathbb{R}^n$ such that for any $t \in U$, the supports of f and g are contained in K. Let $0 < \alpha'' < \alpha < \alpha' < 1$ and $0 < \beta''_l < \beta_l < \beta'_l < 1$, $l = 1, ..., n$.*
Suppose that each f_l, $l = 1, ..., n$ fulfilles (3.15) with α' and β'_l, (3.17) with α' and (3.18) with β'_l.
Suppose further that g satisfies (3.16), $l = 1, ..., n$ with $1 - \alpha''$ and $1 - \beta''_l$, (3.17) with $1 - \alpha''$ and (3.19) with $1 - \beta''_l$, $l = 1, ..., n$.
Then assumptions (I) and (II) are fulfilled.

Proof. We show the assertions for f in the case $n = 1$. First consider II. By standard embedding theorems, $\left\| h | H_p^{\beta'}(\mathbb{R}) \right\|$ is bounded by a constant times

$$\|h|L_p(\mathbb{R})\| + \left(\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|h(x+y) - h(x)|^p}{|y|^{1+\beta' p}} dxdy \right)^{1/p} \tag{3.20}$$

in case $p \leq 2$. For $p > 2$ this has to be replaced by

$$\|h|L_p(\mathbb{R})\| + \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |h(x+y) - h(x)|^p \, dx \right)^{2/p} \frac{dy}{|y|^{1+2\beta'}} \right)^{1/2}, \tag{3.21}$$

see e.g. [78]. For compactly supported h, Hölder's inequality quickly shows that (3.21) is bounded by a constant times (3.20) with $\beta' + \delta$ in place of β' for any $\delta > 0$. Hence it suffices to show that the $L_p((a,b))$-norm of (3.20), with

$$D_{a+}^{\alpha'} f(t,x) = c_{\alpha'} \mathbf{1}_{(a,b)}(t) \left(\frac{f(t,x)}{(t-a)^{\alpha'}} + \alpha' \int_a^t \frac{f(t,x) - f(\tau,x)}{(t-\tau)^{\alpha'+1}} d\tau \right) \tag{3.22}$$

in place of h, is finite. An estimate of the first summand of (3.22) together with the difference part of (3.20) in $L_p((a,b))$ is given by

$$\begin{aligned} &\int_a^b \int \int_{|x-y|<r_0} \left(\int_a^t \frac{|f(t,x) - f(\tau,x) - f(t,y) + f(\tau,y)|}{(t-\tau)^{1+\alpha'}} d\tau \right)^p \times \\ &\qquad\qquad \times \frac{1}{|x-y|^{1+\beta' p}} dxdydt \\ &\leq \int_a^b \int \int_{|x-y|<r_0} \left(\int_a^t |t-\tau|^{(\alpha-\alpha')-1} d\tau \right)^p |x-y|^{(\beta-\beta')p-1} \, dxdydt < \infty, \end{aligned}$$

we have used (2.27). The terms arising from combinations of the remaining summands of (3.22) and (3.20) obey similar estimates obtained from (2.27), (2.30) and the fact that each $f(t,\cdot)$ has compact support.
(I) follows from the continuity of f at a, seen as $H_p^{\alpha'}(\mathbb{R})$-valued function, which is shown by arguments similar to the above.
For g one can proceed similarly, it suffices to note that with $l = 1, ..., n$ fixed,

$\mathbb{R}_l = span\,\{e_l\}$, and $\mathbb{R}_l^\perp \subset \mathbb{R}^n$ denoting its orthogonal complement,

$$\left\| g|H^{\beta_l}_{p',l}(\mathbb{R}^n) \right\| = \left\| \left((1+\xi_l^2)^{\beta_l/2}\hat{g}\right)^\vee |L_{p'}(\mathbb{R}^n) \right\|$$
$$= \left(\int_{\mathbb{R}_l^\perp} \left\| g(\xi_l', \cdot)|H^{\beta_l}_{p'}(\mathbb{R}) \right\|^{p'} d\xi_l' \right)^{1/p'},$$

where $\xi_l' = (\xi_1, ..., \xi_{l-1}, \xi_{l+1}, ..., \xi_n)$. □

Definition 3.3.1. Suppose the functions f and g are as specified above and fulfill (I) and (II). We define the *(hybrid, forward) integral* of f w.r.t. g over $(a,b)\times D$ by

$$\int\int_{(a,b)\times D} \left\langle f, (\frac{\partial}{\partial t}\nabla)^+ g \right\rangle d(t,x) :=$$
$$(-1)^\alpha \int_a^b \int_D \left\langle D^\alpha_{a+} f_{a+}(t), \nabla^+ D^{1-\alpha}_{b-} g_{b-}(t) \right\rangle\, dx\, dt$$
$$+ \int_D \left\langle f(a+), \nabla^+ g(b-) \right\rangle\, dx - \int_D \left\langle f(a+), \nabla^+ g(a+) \right\rangle\, dx, \quad (3.23)$$

the integrals over D with meaning as in (3.10). Here ∇^+ refers to x only.

Remark 3.3.1. The definition is correct, i.e. the value of the integral is independent of the values α and β_l. For β_l this follows from the proof of Proposition 3.2.1, for α it is similar to the one-parameter case, [91].

Proposition 3.3.1. *Under the assumptions of Definition 3.3.1, the integral in (3.23) exists. The spatial forward limit may then be replaced by the corresponding distributional derivative.*

In view of the norms (2.4) and (2.18), this is an immediate consequence of Proposition 3.2.1 together with Hölder's inequality.

3.3.2 Average integrals

We wish to obtain an average version of (3.23), a special case of which will be used to represent the two parameter Itô integral later on. The average integral is obtained by means of averaging with respect to the variable t.

Lemma 3.3.2. *Under the assumptions of Definition 3.3.1, we have*

$$\int_a^b \int_D \left\langle D^{\alpha}_{a+} f(t), \nabla^+ D^{1-\alpha}_{b-} g_{b-}(t) \right\rangle\, dxdt$$
$$= \lim_{\varepsilon\to 0} \int_a^b \int_D \left\langle D^{\alpha-\varepsilon}_{a+} f(t), \nabla^+ D^{1-\alpha}_{b-} g_{b-}(t) \right\rangle\, dxdt.$$

This follows straightforward using Propositions 3.2.1 and 3.3.1 together with the triangle inequality and the continuity properties of fractional integrals.

We now use the additional assumption that

(III) For a.e. $x \in D$, $g(b-, x)$ exists and equals $g(b-)(x)$.

Under the hypotheses of Lemma 3.3.1, (III) is guaranteed.

Lemma 3.3.3. *Let $\varepsilon, r > 0$, $0 < \alpha < 1$ and $0 < \beta < 1/p$. Suppose (I) and (III) are valid, $f_l \in \mathcal{H}^{\alpha-\varepsilon,\beta_l}_{a+,p}$, $l = 1, ..., n$, and $g_{b-} \in \mathcal{H}^{1-\alpha,\overline{1}-\overline{\beta}}_{b-,p'}$. Then*

$$\frac{1-\varepsilon}{\Gamma(\varepsilon)} \int_0^\infty u^{\varepsilon-1} \int_a^b \int_D \left\langle f(t), \nabla^+_r \frac{g_{b-}(t+u) - g_{b-}(t)}{u} \right\rangle dxdtdu$$
$$= (-1)^\alpha \int_a^b \int_D \left\langle D^{\alpha-\varepsilon}_{a+} f(t), \nabla^+_r D^{1-\alpha}_{b-} g_{b-}(t) \right\rangle dxdt\ . \quad (3.24)$$

For the sake of legibility we formulate the proof only for the case $n = 1$ and $D = (a', b')$, in this case the assertion reads

$$\frac{1-\varepsilon}{\Gamma(\varepsilon)} \int_0^\infty u^{\varepsilon-1} \int_a^b \int_{a'}^{b'} f(t,x), \frac{\Delta_{u,r} g_{b-}(t,x)}{ur}\, dxdtdu$$
$$= (-1)^\alpha \int_a^b \int_{a'}^{b'} D^{\alpha-\varepsilon}_{a+} f(t,x), \frac{D^{1-\alpha}_{b-} g_{b-}(t, x+r) - D^{1-\alpha}_{b-} g_{b-}(t,x)}{r}\, dxdt\ . \quad (3.25)$$

The case of general n follows by simple modifications.

Proof. For u and r fixed, we may rewrite the inner integrals on the left hand side of (3.25) as

$$\int_a^b \int_{a'}^{b'} f(t,x) \frac{\Delta_{u,r} g_{b-}(t,x)}{ur}\, dxdt$$
$$= (-1)^{\alpha-\varepsilon} \int_a^b \int_{a'}^{b'} D^{\alpha-\varepsilon}_{a+} f(t,x) \frac{I^{\alpha-\varepsilon}_{b-} \Delta_{u,r} g_{b-}(t,x)}{ur}\, dxdt\ , \quad (3.26)$$

as Fubini's theorem and the integration-by-parts formula for fractional derivatives (2.7) show. Next, note that for $r > 0$ fixed,

$$\frac{(1-\varepsilon)(-1)^{1-\varepsilon}}{\Gamma(\varepsilon)} \int_\delta^\infty \frac{I_{b-}^{\alpha-\varepsilon} \Delta_{u,r} g_{b-}(t,x)}{u^{2-\varepsilon} r} du \ ,$$

seen as Marchaud derivative, converges to

$$-\frac{D_{b-}^{1-\alpha} g_{b-}(t,x+r) - D_{b-}^{1-\alpha} g_{b-}(t,x)}{r} \tag{3.27}$$

as δ decreases to zero. By Fubini's theorem and Hölder's inequality we may therefore rewrite the left hand side of (3.25) as

$$(-1)^\alpha \int_a^b \int_{a'}^{b'} D_{a+}^{\alpha-\varepsilon} f(t,x), \frac{D_{b-}^{1-\alpha} g_{b-}(t,x+r) - D_{b-}^{1-\alpha} g_{b-}(t,x)}{r} \, dxdt \ ,$$

as desired. □

Similar to the former sections we now define an *average integral.* For a function $h : (0,1) \times (0,1) \to \mathbb{R}$ we write $\lim_{|(\varepsilon,r)| \to 0} h(\varepsilon(t), r(t))$ if this limit exists and is independent of the particular path on which (ε, r) tends to the origin. Here $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^2$. Set

$$\begin{aligned}(A) \int_{(a,b)\times D} \left\langle f, (\frac{\partial}{\partial t}\nabla)^+ g \right\rangle \, d(t,x) :=& \\ \lim_{|(\varepsilon,r)|\to 0} \varepsilon \int_0^1 u^{\varepsilon-1} \int_a^b \int_D \left\langle f(t), \nabla_r^+ \frac{g_{b-}(t+u) - g_{b-}(t)}{u} \right\rangle \, dxdtdu \ .& \qquad (3.28)\end{aligned}$$

In view of Lemma 3.3.2 and Lemma 3.3.3, (3.28) is an extension of (3.23).

3.4 Examples involving random fields

Lemma 2.4.1 and the simple cut-off described there can be combined with Lemma 3.3.1. This allows to state the following pathwise result:

Corollary 3.4.1. *Let the vector* $B = (B^{\alpha,\beta_1}, ..., B^{\alpha,\beta_n})$ *consist of* n *hybrid fractional Brownian sheets* B^{α,β_l} *over* $(\Omega, \mathcal{F}, \mathbb{P})$ *with* $0 < \alpha, \beta_l < 1$, $l = 1, ..., n$. *Suppose* $B^{\alpha',\overline{\beta'}}$ *is an anisotropic fractional Brownian sheet over the* *same probability space, such that* $\alpha' > 1-\alpha$ *and* $\beta_l' > 1-\beta_l$, $l = 1, ..., n$. *Then the integral*

$$\int\int_{(a,b)\times D} \left\langle B, (\frac{\partial}{\partial t}\nabla)^{+} B^{\alpha',\overline{\beta'}} \right\rangle d(t,x) \tag{3.29}$$

exists $\mathbb{P}$*-a.s.*

Proof. Let φ be the smooth cut-off function from Remark 2.4.1 and recall that we agreed to write again $B^{\alpha,\beta_l}(\omega)$ to denote $\varphi B^{\alpha,\beta_l}(\omega)$, similarly for $B^{\alpha',\overline{\beta'}}$. Choose some p such that $\beta_l < 1/p$ for all $l = 1, ..., n$. With this notation, $B(\omega) = (B^{\alpha,\beta_1}(\omega), ..., B^{\alpha,\beta_n}(\omega))$ and $B^{\alpha',\overline{\beta'}}(\omega)$ in place of f and g respectively fulfill the hypotheses of Definition 3.3.1 for a.e. $\omega \in \Omega$ by Lemma 3.3.1. For $\mathbb{P}$-a.e. $\omega \in \Omega$, the existence of

$$\int\int_{(a,b)\times D} \left\langle B(\omega), (\frac{\partial}{\partial t}\nabla)^{+} B^{\alpha',\overline{\beta'}}(\omega) \right\rangle d(t,x)$$

follows. The independence of (3.29) of the choice of the function φ is obvious. □

Similar arguments yield a corresponding result for integrals over D only.

3.5 Stochastic integrals as limit cases

We consider stochastic versions of the integrals (3.14) and (3.28), where the integrators are given by the n-parameter Brownian motion and the hybrid Brownian sheet, respectively. For bounded, continuous and predictable integrands H they will be shown to coincide with corresponding integrals of (partial) Itô type.
Again $(\Omega, \mathcal{F}, \mathbb{P})$ is a given probability space. We assume all processes to be real-valued, the extension to complex-valued is straightforward. Throughout this section, D denotes a *bounded and convex domain* in $\mathbb{R}^n$. We remark that under some conditions on the boundary, more general domains can be considered.
We use the identification $x = (x_l', x_l)$, where $x_l' = (x_1, ..., x_{l-1}, x_{l+1}, ..., x_n)$. For any fixed $l = 1, ..., n$ and $x_l' \in \mathbb{R}^{n-1}$, set

$$d_l^+(x_l') := \inf\{r \in \mathbb{R} : (x_l', re_l) \in \partial D\} \ ,$$

$$d_l^-(x_l') := \inf\left\{r > d_l^+(x_l') : (x_l', re_l) \in \partial D\right\}$$

and $\gamma_l^\pm(x_l') := d_l^\pm(x_l')e_l$. Finally, put

$$P_l := P_l(D) := \left\{x_l' \in \mathbb{R}^{n-1} : \exists x_l \in \mathbb{R} \text{ such that } (x_l', x_l) \in D\right\} .$$

As mentioned, setting $\alpha = 1/2$ in (2.24), we arrive at the n-parameter Brownian motion W on $\mathbb{R}^n$. From the covariance structure it follows that for $l = 1, ..., n$ and $x_l' \in P_l$ fixed, $\left\{W^D(x_l', x_l) : x_l \in \mathbb{R}\right\}$ with

$$W^D(x_l', x_l) := W(x_l', x_l) - W(x_l', \gamma_l^+(x_l'))$$

is a Brownian motion on $\mathbb{R}$, $(x_l', \gamma_l^+(x_l'))$ playing the role of the origin. For $l = 1, ..., n$ and $x_l' \in P_l$ fixed, $x \in \mathbb{R}$, $x_l \geq d_l^+(x_l')$, set

$$\mathcal{F}_{x_l}^{x_l'} := \sigma\left(W^D(x_l', y_l) : d_l^+(x_l') \leq y_l \leq x_l\right) .$$

Let $\mathcal{P}^{x_l'}$ denote the σ-field on $[d_l^+(x_l'), \infty) \times \Omega$ generated by integrands of form

$$h(\omega, x_l) = X(\omega)\mathbf{1}_{(y_l,z_l]}(x_l) \quad \text{or} \quad h_0(\omega, x_l) = X_0(\omega)\mathbf{1}_{\left\{d_l^+(x_l')\right\}}(x_l) ,$$

where $\omega \in \Omega$, X is $\mathcal{F}_{y_l}^{x_l'}$-measurable, X_0 is $\mathcal{F}_{\gamma_l^+(x_l')}^{x_l'}$-measurable and $y_l, z_l \in \mathbb{R}$, $d_l^+(x_l') \leq y_l < z_l$. We set $\mathcal{P}^l := \bigcap_{x_l' \in P_l} \mathcal{P}^{x_l'}$ and call a random field $H = (H_1, ..., H_n)$ on $\mathbb{R}^n$ *predictable w.r.t.* D if H_l is $\mathcal{P}^l$-measurable for all $l = 1, ..., n$. For our purposes it seems convenient to assume that $H = (H_1, ..., H_n)$ is defined on the whole of $\mathbb{R}^n$. H is called *square integrable* if for all $l = 1, ..., n$ and all $x_l' \in P_l$, $\mathbb{E}\left\{\int_{d_l^+(x_l')}^{d_l^-(x_l')} H_l(x_l', x_l)^2 dx_l\right\} < \infty$. For predictable and square integrable H, each integral

$$\int_{[d_l^+(x_l')d_l^-(x_l')]} H_l(x_l', x_l) dW^D(x_l', x_l) \ , \ l = 1, ..., n \ , \ x_l' \in P_l \ ,$$

is well defined in the Itô sense and fulfills a corresponding Itô isometry. We obtain a (partial) Itô type integral for such integrands on D by setting

$$\int_{\overline{D}} \langle H, dW \rangle := \sum_{l=1}^n \int_{P_l} \int_{[d_l^+(x_l'), d_l^-(x_l')]} H_l(x_l', x_l) dW_j^D(x_l', x_l) dx_l' \ . \tag{3.30}$$

Assuming in addition that H is bounded and a.s. continuous on $\partial D \subset \mathbb{R}^n$, Proposition 1.1 in [76] shows that the forward integral equals the Itô integral, in our situation that means

$$\int_{[d_l^+(x_l'),d_l^-(x_l')]} H_l(x_l', x_l) dW^D(x_l', x_l)$$
$$= \lim_{r\to 0} \frac{1}{r} \int_{d_l^+(x_l')}^{d_l^-(x_l')} H_l(x_l', x_l) \left(W^D(x_l', x_l + re_l) - W^D(x_l', x_l)\right) \, dx_l \ , \quad (3.31)$$

where the limit is taken in the mean square. The correction denoted by the superscript D may be omitted in the difference, and with the notation of the former sections we consider

$$\int_{\overline{D}} \left\langle H, \nabla_{\mathbb{P}}^+ W \right\rangle dx := \lim_{r\to 0} \int_D \left\langle H, \nabla_r^+ W \right\rangle \, dx \ , \quad (3.32)$$

the limit taken in square mean. We record this result in our formulation:

Lemma 3.5.1. *Let W denote the n-parameter Brownian motion on $\mathbb{R}^n$ and let $H = (H_1, ..., H_n)$ be a bounded random field on $\mathbb{R}^n$, predictable w.r.t. D and a.s. continuous on $\partial D \subset \mathbb{R}^n$. Then the limit (3.32) exists and equals the Itô type integral (3.30),*

$$\int_{\overline{D}} \left\langle H, \nabla_{\mathbb{P}}^+ W \right\rangle dx = \int_{\overline{D}} \langle H, dW \rangle \ .$$

We turn to integrals over $[a, b] \times \overline{D}$.
Setting $\alpha = \beta = 1/2$ in (2.25), we obtain a Gaussian field

$$W = \{W(t, x) : t \in \mathbb{R}, x \in \mathbb{R}^n\} \ ,$$

which might be called the *hybrid Brownian sheet* on $\mathbb{R}^{n+1}$. A corresponding (partial) Itô type integral on $[a, b] \times \overline{D}$ can be defined following the construction of the Itô integral in the plane, see e.g. [15], [85] and [84]:
For $s, t \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$, we write $(t, x) \prec (s, y)$ if and only if $t \le s$ and $x_l \le y_l$, $l = 1, ..., n$. We write $(t, x) \prec\prec (s, y)$ if all inequalities are strict.
Fix some l and $x_l' \in P_l$ as above, then $\left\{W^D(t, x_l', x_l) : t \in \mathbb{R}, x_l \in \mathbb{R}\right\}$ with

$$W^D(t, x_l', x_l) := W(t, x_l', x_l) - W(t, x_l', \gamma_l^+(x_l'))$$

is a two-parameter Brownian sheet and $(0, x_l', \gamma_l^+(x_l'))$ plays the role of the origin.
With l and $x_l' \in P_l$ still fixed we set

$$\mathcal{F}_{t,x_l}^{x_l'} := \sigma\left(W^D(t, x_l', y_l) : (0, d_l^+(x_l')) \prec (s, y_l) \prec (t, x_l)\right).$$

Now let $\mathcal{P}^{x_l'}$ denote the σ-field on $[a, \infty) \times [d_l^+(x_l'), \infty) \times \Omega$ generated by integrands of form

$$h(\omega, t, x_l) = X(\omega)\mathbf{1}_{(c,d]}(t)\mathbf{1}_{(y_l,z_l]}(x_l)\ , \tag{3.33}$$

where $\omega \in \Omega$, X is $\mathcal{F}_{(c;y_l)}^{x_l'}$-measurable, $(a, d_l^+(x_l')) \prec (c, y_l) \prec\prec (z_l, d)$, together with integrands of a similar form but with $\mathbf{1}_{\{0\}}(t)$ in place of $\mathbf{1}_{(c,d]}(t)$ or $\mathbf{1}_{\left\{d_l^+(x_l')\right\}}(t)$ in place of $\mathbf{1}_{(y_l,z_l]}$ or both, considered under the respective measurability assumptions. Put again $\mathcal{P}^l := \bigcap_{x_l' \in P_l} \mathcal{P}^{x_l'}$ and call a random field $H = (H_1, ..., H_n)$ on $\mathbb{R}^{n+1}$ *predictable w.r.t.* $(a, b) \times D$ if H_l is $\mathcal{P}^l$-measurable for all $l = 1, ..., n$. H is called *square integrable* if for all $l = 1, ..., n$ and all $x_l' \in P_l$, $\mathbb{E}\left\{\int\int_{(a,b)\times(d_l^+(x_l'), d_l^-(x_l'))} H_l(t, x_l', x_l)^2 d(t, x_l)\right\} < \infty$. For predictable and square integrable H, each integral

$$\int\int_{[a,b]\times[d_l^+(x_l'), d_l^-(x_l')]} H_l(t, x_l', x_l) dW(t, x_l', x_l) \tag{3.34}$$

is well defined as *two-parameter Itô integral (of the first kind)*. For integrands of form (3.33) it equals $X\Delta_{d-c, z_l - y_l} W(c, x_l', y_l)$, the difference according to (2.26) referring to c and y_l. The usual isometry property holds at the level of (3.34). Setting

$$\int\int_{[a,b]\times\overline{D}} \langle H, dW\rangle := \sum_{l=1}^n \int_{P_l} \int\int_{[a,b]\times[d_l^+(x_l'), d_l^-(x_l')]} H_l(t, x_l', x_l) dW(t, x_l', x_l) dx_l'\ , \tag{3.35}$$

we obtain an integral of (partial) Itô type.
Now consider the limit

$$\lim_{|(u,r)|\to 0} \int_a^b \int_D \left\langle H(t, x), \nabla_r^+ \frac{W_b(t+u, x) - W_b(t, x)}{ur}\right\rangle dxdt\ , \tag{3.36}$$

taken in the mean square and, whenever it exists. The correction $W_b(t, x) = \mathbf{1}_{(a,b)}(t)\left(W(t, x) - W(b, x)\right)$ w.r.t. (a, b) is understood pathwise. For $u, r > 0$

fixed, a member of the sequence in (3.36) equals

$$\sum_{l=1}^{n} \int \int_{(a,b)\times D} H_l(t,x) \frac{\Delta_{u,re_l} W_b(t;x)}{ur} \, dtdx \ ,$$

with $\Delta_{u,re_l} W_b(t,x)$ according to (2.26). Consider its average version

$$(A) \int_{[a,b]\times \overline{D}} \left\langle H, (\frac{\partial}{\partial t}\nabla)_{\mathbb{P}}^{+} W \right\rangle \, d(t,x) :=$$
$$\lim_{|(\varepsilon,r)|\to 0} \varepsilon \int_0^1 u^{\varepsilon-1} \int_a^b \int_D \left\langle H(t), \nabla_r^{+} \frac{W_b(t+u) - W_b(t)}{u} \right\rangle \, dxdtdu \ , \quad (3.37)$$

taken in square mean, whenever it exists. This is the *stochastic variant of the pathwise (hybrid) average integral* (3.28). Adapting the proofs known from the one-parameter case, [76], [93], we see that for suitable integrands, it equals the Itô type integral:

Lemma 3.5.2. *Let $H = (H_1, ..., H_n)$ be a random field on $\mathbb{R}^{n+1}$, predictable w.r.t. $[a,b] \times \overline{D}$, a.s. bounded and continuous. Then:*

(i) If limit (3.36) exists, so does the average limit (3.37), and both agree.

(ii) The average limit (3.36) exists and equals the Itô-type integral (3.35).

Consequently also

$$(A) \int_{[a,b]\times \overline{D}} \left\langle H, (\frac{\partial}{\partial t}\nabla)_{\mathbb{P}}^{+} W \right\rangle \, d(t,x) = \int \int_{[a,b]\times \overline{D}} \langle H, dW \rangle \ .$$

(i) is an obvious modification of arguments from [93], p.3. For convenience, the proof of (ii) is sketched for the case $n = 1$, $D = (a', b')$, where W^D is the Brownian sheet on $\mathbb{R}^2$.

Proof. Notice first that if (3.36) exists with $W_{b,b'}(t,x) = W(t,x) - W(b,x) - W(t,b') + W(b,b')$ in place of $W_b(t,x)$, it exists in its original form. This follows using bounded convergence. Now follow [76] : For any $u, u > 0$ and any $(t,x) \in [a,b] \times [a', b']$,

$$\Delta_{u,r} W_{b,b'}(t,x)$$
$$= W((t+u)\wedge b, (x+r)\wedge b') - W(t, (x+r)\wedge b') - W((t+u)\wedge b, x) + W(t,x) \ .$$

Now it follows that with $Y(t,x) := H(t,x)\mathbf{1}_{\{(t,x)\prec\prec(s,y)\prec(t+u,x+r)\}}$,

$$\int\int_{[a,b]\times[a',b']} Y dW = H(t,x)\Delta_{u,r}W_{b,b'}(t,x) \ ,$$

the right hand side in the Itô sense. By Lemma 3.5.3 below then

$$\frac{1}{ur}\int_{[a,b]\times[a',b']} H(t,x)\Delta_{u,r}W_{b,b'}(t,x)d(t,x) = \int\int_{[a,b]\times[a',b']} \times$$
$$\left(\frac{1}{ur}\int_{[s-u,s]\times[y-r,y]} H(t,x)\mathbf{1}_{[a,b]\times[a',b']}(t,x)d(t,x)\right)\mathbf{1}_{[a,b]\times[a',b']}(s,y)dW(s,y) \ .$$

By the isometry, the expectation of the square of

$$\int\int_{[a,b]\times[a',b']} \times$$
$$\left(\frac{1}{ur}\int\int_{[s-u,s]\times[y-r,y]} H(t,x)\mathbf{1}_{[a,b]\times[a',b']}(t,x)d(t,x) - H(s,y)\right) dW(s,y)$$

equals the expectation of

$$\int\int_{[a,b]\times[a',b']} \times$$
$$\left(\frac{1}{ur}\int\int_{[s-u,s]\times[y-r,y]} H(t,x)\mathbf{1}_{[a,b]\times[a',b']}(t,x)d(t,x) - H(s,y)\right)^2 d(s,y) \ .$$

Now take into account that along any increasing path (u,r),

$$\lim_{|(u,r)|\to 0} \frac{1}{ur}\int_{[s-u,s]\times[y-r,y]} H(t,x)\mathbf{1}_{[a,b]\times[a',b']}(t,x)d(t,x) = H(s,y) \quad \text{a.s.}$$

by the a.s. continuity of H. □

Lemma 3.5.3. *Let H be a bounded and $\mathcal{P}\otimes\mathcal{B}([a,b]\times[a',b'])$-measurable mapping on $\Omega\times[a,b]\times[a',b']\times[a,b]\times[a',b']$. Then*

$$\int_{[a,b]\times[a',b']}\int_{[a,b]\times[a',b']} H(u,v)dW(u)dv = \int_{[a,b]\times[a',b']}\int_{[a,b]\times[a',b']} H(u,v)dvdW(u) \ .$$

For integrands of form (3.33) this is obvious, for arbitrary H it follows by a monotone class argument, use the isometry together with the bounded convergence theorem.

Remark 3.5.1. In the case $n = 1$, [84] used a weakly adapted two-parameter integral to study partial differential equations, contained in a construction similar to the above using the filtrations $\mathcal{F}_t^{x'_l} := \bigvee_{d_l^+(x'_l) \leq x_l} \mathcal{F}_{t,x_l}^{x'_l}$. The above can easily be adapted to this case.

Chapter 4

Partial differential equations

4.1 Pathwise integral operators

Let $k, n \in \mathbb{N} \setminus \{0\}$. $D \subset \mathbb{R}^n$ is a bounded C^∞-domain.

We specify the driving field. Let $Z = (Z^1, ..., Z^k)$, $Z^j = Z^j(t; x_1, .., x_n)$, $j = 1, ..., k$, be an $\mathbb{R}^k$-valued vector field on $\mathbb{R}^{n+1}$.
Below we will consider Z also as Banach space valued function $Z(t)$ of the time parameter t. In this case we put

$$\begin{aligned} Z_t^j(s) &:= \mathbf{1}_{(0,t)}(s) \left(Z^j(s) - Z^j(t) \right) \quad \text{and} \\ Z_t(s) &:= \mathbf{1}_{(0,t)}(s) \left(Z(s) - Z(t) \right) \ . \end{aligned} \tag{4.1}$$

The values $Z^j(t)$ will be assumed to exist for each $t > 0$ in the pointwise sense. We do not assume Z to be differentiable.

In the following, we motivate and state a *formal definition* of the operator that realizes integration with respect to Z.

Recall that $D_{t-}^{1-\alpha}$ denotes the *(right-sided) Weyl-Marchaud fractional derivative operator of order* $0 < \alpha < 1$ on $(0, t)$, $t > 0$.
For the field Z as above, it will follow from the hypotheses below that seen as vector valued functions, the fractional derivatives $s \mapsto D_{t-}^{1-\alpha} Z_t(s)$ take values in some Sobolev space contained in a scale of spaces. Within this scale, the fractional powers A^α of A act as isomorphisms and the Sobolev spaces appear as domains of definition of the A^α or their duals. For a certain range of α,

the analytic semigroup $(P(t))_{t\geq 0}$ generated by $-A$ may then be applied to distributions. We had already pointed out this fact in Subsection 2.3.4.

In Subsection 3.1.4 we had briefly discussed how a suitable integral might look like. Here we give a *heuristic motivation* for the definition of the integral operator below:
Assume $k=1$ and recall Definition 3.3.1. Let $p(t,x,y)$ denote the transition densities corresponding to the heat semigroup $(P(t))_{t\geq 0}$, i.e. $P(t)f(x) = \int_D p(t,x,y)f(y)dx$, and assume for a moment they are regular enough to write

$$(-1)^{\alpha}\int_{(0,t)}\int_D \left\langle D^{\alpha}_{0+}\psi(s), \nabla D^{1-\alpha}_{t-}Z_t(s)\right\rangle dyds \ , \tag{4.2}$$

where $\psi(s) = p(t-s,x,\cdot)g(s,\cdot)$, $g=(g_1,...,g_n)$, $g_l = g_l(s,y)$, denotes a $\mathbb{R}^n$-valued field on $\mathbb{R}^{n+1}$ and $0<\alpha<1$. That means, we integrate $p(t-s,x,y)g(s,y) = (p(t-s,x,y)g_1(s,y),...,p(t-s,x,y)g_n(s,y))$ with respect to $Z(s,y)$ using the two-parameter integral from Section 3.3.
Taking into account the definition of D^{α}_{0+}, carrying out the integration over D and rearranging the terms, (4.2) is seen to be the sum over $l=1,...,n$ of

$$\begin{aligned}
&\frac{(-1)^{\alpha}}{\Gamma(1-\alpha)}\int_0^t s^{-\alpha}P(t-s)\left(g_l(s)\frac{\partial}{\partial y_l}D^{1-\alpha}_{t-}Z_t(s)\right)ds \\
&+\frac{\alpha(-1)^{\alpha}}{\Gamma(1-\alpha)}\int_0^t\int_0^s (s-\sigma)^{-\alpha-1}(P(t-s)-P(t-\sigma))\left(g_l(s)\frac{\partial}{\partial y_l}D^{1-\alpha}_{t}Z_t(s)\right)d\sigma ds \\
&+\frac{\alpha(-1)^{\alpha}}{\Gamma(1-\alpha)}\int_0^t\int_0^s (s-\sigma)^{-\alpha-1}P(t-\sigma)\left((g_l(s)-g_l(\sigma))\frac{\partial}{\partial y_l}D^{1-\alpha}_{t}Z_t(s)\right)d\sigma ds \ .
\end{aligned} \tag{4.3}$$

It is convenient to express the middle summand of (4.3) in terms of fractional powers of A. We use the semigroup property together with the analyticity and the fact that

$$\int_0^s u^{-\alpha-1}(I-P(u))f du = \frac{\Gamma(1-\alpha)}{\alpha}A^{\alpha}f - \frac{1}{\alpha}s^{-\alpha}f + \int_s^{\infty} u^{-\alpha-1}P(u)f du \ ,$$

for $f\in dom(A^{\alpha})$, where I is the identity operator. Inserting this into (4.3), the term arising from the summand $\alpha^{-1}s^{-\alpha}f$ cancels with the first summand in (4.3), and we arrive at the expression in Definition 4.1.1 below.
In the former definitions we would have corrected the integrand $p(t-s,x,\cdot)g_l(s)$

at $s = 0$ and added the correction terms $P(t)\left(g_l(0)\frac{\partial}{\partial y_l}(Z(t) - Z(0))\right)$. Here these corrections cancel and may be omitted.
Now the formulation involves the semigroup operators themselves, which are easier to handle than their transition densities - for the latter the precise regularity in time is hardly known, for the semigroup we can use many properties that stem from its analyticity.

This leads to the following *rigorous definition.*
Let $k \in \mathbb{N} \setminus \{0\}$ and suppose that either $g = (g_l^j)_{\substack{l=1,..,n \\ j=1,...,k}}$ is a constant real $(n \times k)$-matrix, $g \in \mathcal{M}(n \times k, \mathbb{R})$ or, g is an $\mathcal{M}(n \times k, \mathbb{R})$-valued field on $\mathbb{R}^{n+1}$, such that all rows $g_l = (g_l^1, ..., g_l^k)$ with $g_l^j = g_l^j(t; x_1, ..., x_n)$, seen as vector valued functions $t \mapsto g_l(t)$, also admit their values in a Sobolev space contained in that scale.
The gradient is taken in distributional sense and always refers to the space variable $x = (x_1, ..., x_n)$. We use the notation (2.1).

Definition 4.1.1. Let $0 < \alpha < 1$. For $t > 0$, set

$$\begin{aligned} I_t^\alpha\left(g, \frac{\partial}{\partial t}\nabla Z\right) &:= (-1)^\alpha \int_0^t A^\alpha P(t-s)\left\langle g(s), \nabla D_{t-}^{1-\alpha} Z_t(s)\right\rangle ds \\ &+ c_\alpha(-1)^\alpha \int_0^t \int_0^s (s-\sigma)^{-\alpha-1} P(t-\sigma)\left\langle (g(s) - g(\sigma)), \nabla D_{t-}^{1-\alpha} Z_t(s)\right\rangle d\sigma ds \\ &+ c_\alpha(-1)^\alpha \int_0^t \int_s^\infty \sigma^{-\alpha-1} P(\sigma + t - s)\left\langle g(s), \nabla D_{t-}^{1-\alpha} Z_t(s)\right\rangle d\sigma ds\ . \end{aligned} \tag{4.4}$$

The number c_α is given by $c_\alpha = \alpha\Gamma(1-\alpha)^{-1}$.

Each semigroup operator applies to the entire k-valued term in sharp brackets.
The integral terms contain products of functions and distributions. We define them by means of *paraproducts* as studied in [79] and [75], see Section 2.3, especially Lemma 2.3.1. This notion includes the product definitions used in the concrete examples of [56] and [36].

Later on it will be shown that Definition 4.1.1 is *correct* in the following sense:
Under the respective hypotheses of Theorems 4.2.1, 4.3.1 and 4.5.1 below, $I_t^\alpha\left(g, \frac{\partial}{\partial t}\nabla Z\right)$ *exists and does not depend on the particular choice of* α. *In the*

following, we write $I_t\left(g, \frac{\partial}{\partial t}\nabla Z\right)$.

4.2 Systems with linear multiplicative noise

Linear systems under additive noises allow some refined assertions, therefore they are considered later.
We start with a study of *systems of semilinear parabolic equations with linear multiplicative gradient type noise.*

4.2.1 The problem

Let $G = (G_1, ..., G_n)$ denote an $\mathcal{M}(n \times k, \mathbb{R})$-valued field on $\mathbb{R}^k$, such that each row $G_l = (G_l^1, ..., G_l^k) : \mathbb{R}^k \to \mathbb{R}^k$ is a linear mapping.

Let $T > 0$ be arbitrary. Consider the formal problem given by

$$\frac{\partial u}{\partial t}(t,x) = (-Au)(t,x) + \left\langle G(u), \frac{\partial}{\partial t}\nabla Z \right\rangle (t,x) \ , \tag{4.5}$$

$t \in (0,T),\ x \in D$, together with the *Dirichlet boundary condition*

$$u(\cdot,t)|_{\partial D} = 0 \ , \ t \in (0,T) \ , \tag{4.6}$$

and with *initial condition*

$$u(0,x) = f(x) \ , \ x \in D \ . \tag{4.7}$$

By (2.1) and (2.2), we formally have $\left\langle G(u), \frac{\partial}{\partial t}\nabla Z\right\rangle = \sum_{l=1}^n G_l(u) \cdot \frac{\partial^2 Z}{\partial t \partial x_l}$.

The problem (4.5)-(4.7) is made rigorous in the sense of mild solutions:

Definition 4.2.1. A function u is a *(mild) solution* to (4.5)-(4.7) if it satisfies the integral equation

$$u(t) = P(t)f + I_t\left(G(u), \frac{\partial}{\partial t}\nabla Z\right) \quad , \quad t \in (0,T) \ . \tag{4.8}$$

Equation (4.5) allows to describe diffusion phenomena under couplings caused by the cross diffusion term Au or the noise term $\left\langle G(u), \frac{\partial}{\partial t}\nabla Z\right\rangle$.

4.2.2 Existence and uniqueness of solutions

For $k \in \mathbb{N}$, $k \geq 1$ and $0 < \gamma < 1$, $\delta \in \mathbb{R}$ and $1 < p < \infty$, $C^{\gamma}([0,T], H_p^{\delta}(\mathbb{R}^n, \mathbb{R}^k))$ denotes the space of γ-Hölder continuous $H_p^{\delta}(\mathbb{R}^n, \mathbb{R}^k)$-valued functions on $[0,T]$, such that

$$\left\| u | C^{\gamma}([0,T], H_p^{1-\beta}(\mathbb{R}^n, \mathbb{R}^k)) \right\| := \sup_{0 \leq \tau < t \leq T} \frac{\left\| u(t) - u(\tau) | H_p^{\delta}(\mathbb{R}^n, \mathbb{R}^k) \right\|}{(t-\tau)^{\gamma}} < \infty \ . \tag{4.9}$$

Partly following [56], we denote by $W^{\gamma}([0,T], \mathring{H}_2^{\delta}(D, \mathbb{R}^k))$, $0 < \gamma < 1$, $\delta \in \mathbb{R}$, the space of $\mathring{H}_2^{\delta}(D, \mathbb{R}^k)$-valued functions on $[0,T]$ such that

$$\|u\|_{\gamma,\delta} := \sup_{0 \leq t \leq T} \left(\|u(t)\|_{\delta} + \int_0^t \frac{\|u(t) - u(\tau)\|_{\delta}}{(t-\tau)^{\gamma+1}} d\tau \right) < \infty, \tag{4.10}$$

recall that $\|\cdot\|_{\delta}$ denotes the norm in $H_2^{\delta}(\mathbb{R}^n, \mathbb{R}^k)$. We use the above shortcut notation for this norm to simplify the reading of the proofs below.

Theorem 4.2.1. *Suppose $0 < \alpha, \beta, \gamma, \varepsilon < 1$ and $Z = (Z^1, ..., Z^k)$ is a vector field on $\mathbb{R}^{n+1}$ such that $Z \in C^{1-\alpha}([0,T], H_q^{1-\beta}(\mathbb{R}^n, \mathbb{R}^k))$ for some $q > 2 \vee (n/\delta)$.*
Let G be as specified above, and assume $f \in \mathring{H}_2^{\delta+2\gamma+\varepsilon}(D, \mathbb{R}^k)$, $2\gamma + \delta + \varepsilon < 3/2$. Further, let

$$\alpha < \gamma < 1 - \alpha \quad , \quad \beta < \delta \quad \text{and} \quad 2\gamma + \delta < 2 - 2\alpha - \beta \ .$$

Then (4.5)-(4.7) has a unique solution u in the space $W^{\gamma}([0,T], \mathring{H}_2^{\delta}(D, \mathbb{R}^k))$.

A detailed proof is given in Chapter 5 below.

Remark 4.2.1.

(i) $1 - \alpha$ and $1 - \beta$ describe the temporal and spatial regularity of the driving field Z, respectively. γ and δ are the temporal and spatial regularity parameters of the solution u.
In particular, the theorem requires $\alpha < 1/2$, in other words, a *temporal regularity of the driving that is greater* $1/2$.

(ii) The smoothness assumptions on the initial condition is imposed only to show $P(\cdot)f$ is a member of $W^{\gamma}([0,T], \mathring{H}_2^{\delta}(D, \mathbb{R}^k))$. It is not directly related to the integral construction.

(iii) The choice of the integrability parameter $q > 2 \vee (n/\delta)$ ensures that Lemma 2.3.1 is applicable in order to evaluate the occuring product. As our main applications involve Hölder continuous driving fields Z on compact spatial domains, this is not a major restriction.

4.3 Systems with non-linear multiplicative noise

Under familiar dimension conditions, the result can be generalized to *systems with non-linear coupling and non-linear multiplicative gradient type noise term*.

4.3.1 The problem

Let $F : \mathbb{R}^k \to \mathbb{R}^k$ be a C^1-mapping such that $F(0) = 0$ and having bounded differential $\mathcal{D}F$. That means, if $\|\cdot\|_{L(\mathbb{R}^k,\mathbb{R}^k)}$ denotes a norm in $L(\mathbb{R}^k, \mathbb{R}^k)$, we have

$$\sup_{x\in\mathbb{R}^k} \|\mathcal{D}F(x)\|_{L(\mathbb{R}^k,\mathbb{R}^k)} < M \tag{4.11}$$

for some number $M > 0$.

Let $G = (G_1, ..., G_n)$ denote an $\mathcal{M}(n \times k, \mathbb{R})$-valued field on $\mathbb{R}^k$ such that each $G_l = (G_l^1, ..., G_l^k) : \mathbb{R}^k \to \mathbb{R}^k$ is a C^2-mapping, which fulfilles $G_l(0) = 0$ and has a second order differential $\mathcal{D}^2 G_l$, which is bounded and Lipschitz continuous. That means, if $\|\cdot\|_{L(\mathbb{R}^k,L(\mathbb{R}^k,\mathbb{R}^k))}$ is a norm in $L(\mathbb{R}^k, L(\mathbb{R}^k, \mathbb{R}^k))$, we have

$$\sup_{x\in\mathbb{R}^k} \left\|\mathcal{D}^2 G_l(x)\right\|_{L(\mathbb{R}^k,L(\mathbb{R}^k,\mathbb{R}^k))} < M \tag{4.12}$$

and

$$\left\|\mathcal{D}^2 G_l(x) - \mathcal{D}^2 G_l(y)\right\|_{L(\mathbb{R}^k,L(\mathbb{R}^k,\mathbb{R}^k))} \leq L\,|x-y|_k \quad , \quad x, y \in \mathbb{R}^k \ , \tag{4.13}$$

with some numbers $M, L > 0$.
If, for example, each G_l is a compactly supported C^∞-mapping, these properties are obvious.

Consider the formal problem given by

$$\frac{\partial u}{\partial t}(t,x) = (-Au)\,(t,x) + F(u(t,x)) + \left\langle G(u), \frac{\partial}{\partial t}\nabla Z\right\rangle (t,x) \ , \tag{4.14}$$

$t \in (0,T)$, $x \in D$, together with the former boundary and initial conditions (4.6) and (4.7).

Definition 4.3.1. A function u is called a (mild) solution to (4.14), (4.6) and (4.7) if it satisfies

$$u(t) = P(t)f + \int_0^t P(t-s)F(u(s))ds + I_t\left(G(u), \frac{\partial}{\partial t}\nabla Z\right) \quad , \quad t \in (0,T) \ .$$

F and G act as composition operators, recall Subsection 2.3.6. As usual, their Fréchet derivatives are used to derive key estimates. This involves pointwise multiplication in a single H_2^δ-space, what in turn forces to restrict to the subspace consisting of L_∞-functions. In this case, the subspace under consideration is a multiplication algebra, cf. Section 2.3.

4.3.2 Existence and uniqueness of solutions

Denote by $W^\gamma([0,T], \mathring{H}^\delta_{2,\infty}(D,\mathbb{R}^k))$, $0<\gamma,\delta<1$, the space of $\mathring{H}^\delta_{2,\infty}(D,\mathbb{R}^k) = \mathring{H}^\delta_2(D,\mathbb{R}^k) \cap L_\infty(\mathbb{R}^n,\mathbb{R}^k)$-valued functions on $[0,T]$ such that

$$\|u\|_{\gamma,\delta,\infty} := \sup_{0\le t\le T}\left(\|u(t)\|_{\delta,\infty} + \int_0^t \frac{\|u(t)-u(\tau)\|_{\delta,\infty}}{(t-\tau)^{\gamma+1}}d\tau\right) < \infty. \qquad (4.15)$$

Here $\|\cdot\|_{\delta,\infty} := \|\cdot\|_\delta + \|\cdot\|_\infty$, where $\|\cdot\|_\delta$ is the norm in $H_2^\delta(\mathbb{R}^n,\mathbb{R}^k)$ and $\|\cdot\|_\infty$ that in $L_\infty(\mathbb{R}^n,\mathbb{R}^k)$. We obtain:

Theorem 4.3.1. *Let $0<\alpha,\beta,\gamma,\delta,\varepsilon<1$ and $Z \in C^{1-\alpha}([0,T], H_q^{1-\beta}(\mathbb{R}^n,\mathbb{R}^k))$ for some $q > 2 \vee (n/\delta)$.*
Let F and G be as specified above, in particular, let (4.11), (4.12) and (4.13) be fulfilled. Assume $f \in \mathring{H}^{\delta+2\gamma+\varepsilon}_{2,\infty}(D,\mathbb{R}^k)$, $2\gamma+\delta+\varepsilon<3/2$.
Further, assume

$$\alpha < \gamma < 1-\alpha \quad , \quad \beta < \delta \quad , \quad \text{and} \quad 2\gamma + \delta \vee (n/2) < 2 - 2\alpha - \beta \ .$$

Then (4.14), (4.6), (4.7) has a unique solution u in $W^\gamma([0,T], \mathring{H}^\delta_{2,\infty}(D,\mathbb{R}^k))$.

The proof is discussed in Chapter 5.

Remark 4.3.1. Note that apart from $\alpha < 1/2$, Theorem 4.3.1 forces another restriction on the temporal regularity of the driving field Z. Only if $n/4 < 1-\alpha$ we can find some $0<\beta<1$, such that Theorem 4.3.1 guarantees the existence and uniqueness of a function solution to the non-linear problem (4.14), (4.6), (4.7). This bound had already appeared in [56].

4.4 Some remarks

We can refine our hypotheses on Z. Comparing (4.5) and the proof of Theorem 4.2.1, respectively the key estimate in Proposition 5.1.2 below, we observe that if $G^j_l \equiv 0$, the term $\frac{\partial}{\partial x_l} D^{1-\alpha}_{t-}(Z^j)_t$ has no influence on u. This allows lower (non-negative) degrees of spatial smoothness of Z^j in these directions e_l. Now let $G^j = (G^j_1, ..., G^j_n)$, $j = 1, ..., k$, denote the columns of G.

We define the space $C^\gamma([0,T], H^\delta_{p,G^j}(\mathbb{R}^n, \mathbb{R}))$, $1 < p < \infty$, $0 < \gamma < 1$, $\delta \in \mathbb{R}$, of γ-Hölder continuous $H^\delta_{p,G^j}(\mathbb{R}^n, \mathbb{R})$-valued functions on $[0,T]$ by

$$\left\|u|C^\gamma([0,T], H^\delta_{p,G^j}(\mathbb{R}^n, \mathbb{R}))\right\| := \sup_{0\le\tau<t\le T} \frac{\left\|u(t) - u(\tau)|H^\delta_{p,G^j}(\mathbb{R}^n, \mathbb{R})\right\|}{(t-\tau)^\gamma} < \infty \ , \tag{4.16}$$

where $H^\delta_{p,G^j}(\mathbb{R}^n, \mathbb{R})$ is defined according to Subsection 2.3.2. We immediately obtain:

Corollary 4.4.1. *The assertions of Theorems 4.2.1 and 4.3.1 remain valid if the hypotheses on Z there are replaced by $Z^j \in C^{1-\alpha}([0,T], H^{1-\beta}_{q,G^j}(\mathbb{R}^n, \mathbb{R}))$, $j = 1, ..., k$, $q > 2 \vee (n/\delta)$.*

Apart from these refinements, we wish to point out some further interesting facts:

Remark 4.4.1. For $T' > T > 0$, the space $W^\gamma([0,T'], \mathring{H}^\delta_2(D, \mathbb{R}^k))$ is continuously embedded into $W^\gamma([0,T], \mathring{H}^\delta_2(D, \mathbb{R}^k))$.
Hence, if a function u is the unique solution on $[0,T']$ to the problem associated with (4.5) or (4.14), it must coincide on $[0,T]$ with the unique solution obtained as the fixed point in the space $W^\gamma([0,T], \mathring{H}^\delta_2(D, \mathbb{R}^k))$. This *consistency property* yields a unique solution $u = (u(t) : t \in [0,\infty))$ of the corresponding problem. Similarly for the versions with spatially bounded solutions.
We remark in advance that in the case of suitable random Z, we may assume that there is a common probability null set such that on its complement, u solves the respective problem for any time horizon in the pathwise sense.

Remark 4.4.2. In view of the facts listed in Section 2.3, we might as well treat boundary initial value problems in general $L_p(D, \mathbb{R}^k)$-spaces, $1 < p <$

∞. In this case, the operator A in (2.22) needs to be chosen carefully. We may for instance take the matrix B to be diagonal and choose A_0 to be a positive operator with bounded imaginary powers and domain $H^2_{p,0}(D) := \left\{f \in H^2_p(D) : f|_{\partial\Omega} = 0\right\}$ in $L_p(D)$. We refer to [81], in particular to 4.9.1.

4.5 Linear systems with additive noise

In the case of *linear systems under additive noise*, the smoothness of the solution can be measured in a Hölder norm with respect to the time variable. This is slightly stronger than the formulations above.

4.5.1 The problem

Suppose the term $G(u)$ is replaced by a constant matrix $G = (G^j_l) \in \mathcal{M}(n \times k, \mathbb{R})$ and consider the formal equation

$$\frac{\partial u}{\partial t}(t,x) = (-Au)(t,x) + \left\langle G, \frac{\partial}{\partial t}\nabla Z\right\rangle (t,x) \ , \tag{4.17}$$

$t \in (0,T),\ x \in D$. The equation (4.17) is linear, the noise is additive, and the middle summand in (4.4) vanishes.

Definition 4.5.1. u solves the problem (4.17), (4.6), (4.7) if

$$u(t) = P(t)f + I_t\left(G, \frac{\partial}{\partial t}\nabla Z\right) \quad , \quad t \in (0,T) \ . \tag{4.18}$$

4.5.2 Existence of solutions

For $0 < \gamma < 1$ and $\delta \in \mathbb{R}$, let $C^\gamma([0,T], \mathring{H}^\delta_2(D,\mathbb{R}^k))$ be the space of γ-Hölder continuous $\mathring{H}^\delta_2(D,\mathbb{R}^k)$-valued functions on $[0,T]$, such that

$$\left\| u | C^\gamma([0,T], H^\delta_2(D,\mathbb{R}^k))\right\| < \infty \ .$$

Also the proof of the following is contained in Chapter 5:

Theorem 4.5.1. *Suppose* $0 < \alpha, \beta, \gamma, \varepsilon < 1$, $\delta > 0$ *and* $Z \in C^{1-\alpha}([0,T], H^{1-\beta}_2(\mathbb{R}^n,\mathbb{R}^k))$. *Let* $G = (G^j_l) \in \mathcal{M}(k \times n, \mathbb{R})$*, and assume that* $f \in \mathring{H}^{\delta+2\gamma+\varepsilon}_2(D,\mathbb{R}^k)$*,* $2\gamma + \delta + \varepsilon < 3/2$.

Then the solution u according to (4.8) exists and is in $C^\gamma([0,T], \mathring{H}_2^\delta(D,\mathbb{R}^k))$, provided

$$\gamma < 1-\alpha \quad \text{and} \quad 2\gamma+\delta < 2-2\alpha-\beta \ .$$

In the linear case, uniqueness is immediate from the definition.
Note that for any $\varepsilon > 0$, $C^{\gamma+\varepsilon}([0,T], \mathring{H}_2^\delta(D,\mathbb{R}^k))$ is continuously embedded in $W^\gamma([0,T], \mathring{H}_2^\delta(D,\mathbb{R}^k))$.
Note also that function solutions can be found for arbitrary $\alpha \in (0,1)$.

Obviously the hypotheses can be refined as before, we omit it.

4.6 Applications involving random fields

Recall the notions from Section 2.4. In particular we had collected some Hölder conditions, we repeat them briefly for convenience:

$$|\Delta_{u,r} z(t,x)| \leq c|u|^{\alpha'}|r|_n^{\beta'} \ , \tag{4.19}$$

$$|\Delta_{u,se_l} z(t,x)| \leq c|u|^{\alpha'}|s|^{\beta_l'} \ , \quad l=1,...,n \ , \tag{4.20}$$

$$|z(t+u,x)-z(t,x)| \leq c|u|^{\alpha'} \ , \tag{4.21}$$

$$|z(t,x+r)-z(t,x)| \leq c|r|_n^{\beta'} \ , \tag{4.22}$$

$$|z(t,x+se_l)-z(t,x)| \leq c|s|^{\beta_l'} \ , \quad l=1,...,n \ . \tag{4.23}$$

z is some function of a one-dimensional parameter t and an n-dimensional parameter x. For the precise formulation, see Section 2.4.

In the following, $g=(g_1,...,g_n)$ denotes an index vector for partial potential spaces as described in Sections 2.3 and 4.4.

Lemma 4.6.1. *Let $z:\mathbb{R}^{n+1}\to\mathbb{R}$ be a function, let $U\subset\mathbb{R}$ be an open neighbourhood of $[0,T]$ and assume there is a compact set $K\subset\mathbb{R}^n$ such that for all $t\in U$, the support of $z(t)=z(t,\cdot):\mathbb{R}^n\to\mathbb{R}$ is contained in K. Let $1<q<\infty$.*

(i) If z fulfilles the Hölder conditions (4.19), (4.21) and (4.22) for any $(t,x)\in[0,T]\times K$ with exponents $0<\alpha<1$ and $0<\beta<1$ (in place of α' and β' there), then $z\in C^{\alpha'}([0,T],H_q^{\beta'}(\mathbb{R}^n,\mathbb{R}))$, provided $0<\alpha'<\alpha$ and $0<\beta'<\beta$.

(ii) *Let $g = (g_1, ..., g_n)$ be an index vector. If z fulfilles the Hölder conditions (4.20), (4.21) and (4.23) for any $(t, x) \in [0, T] \times K$ with exponents $0 < \alpha < 1$ and $0 < \beta_l < 1$ (in place of α' and β_l' there), then $z \in C^{\alpha'}([0, T], H_{q,g}^{\beta'}(\mathbb{R}^n, \mathbb{R}))$, provided $0 < \alpha' < \alpha$ and $0 < \beta_l' < \beta_l$ for all l such that $g_l \neq 0$.*

The proof is similar to that of Lemma 3.3.1 and therefore omitted.

Now recall Remark 2.4.1 and the convention to write $B^{\alpha,\beta}(\omega)$ and $B^{\alpha,\overline{\beta}}(\omega)$ for the products $\varphi B^{\alpha,\beta}(\omega)$ and $\varphi B^{\alpha,\overline{\beta}}(\omega)$ of the paths and the cut-off function φ specified there. From Lemma 2.4.1 and the preceding result applied to $B^{\alpha,\beta}(\omega)$ or $B^{\alpha,\overline{\beta}}(\omega)$ in place of z we obtain:

Corollary 4.6.1. *Let $0 < \alpha, \beta, \beta_l < 1$, $l = 1, ..., n$, $\overline{\beta} = (\beta_1, ..., \beta_n)$, $1 < q < \infty$.*

(i) *Let $g = (g_1, ..., g_n)$ be an index vector. The anisotropic fractional Brownian sheet $B^{\alpha,\overline{\beta}}$ possesses a modification, again denoted by $B^{\alpha,\overline{\beta}}$, such that (with the above convention) for $\mathbb{P}$-a.e. $\omega \in \Omega$, $B^{\alpha,\overline{\beta}}(\omega) \in C^{\alpha'}([0, T], H_{q,g}^{\beta'}(\mathbb{R}^n, \mathbb{R}))$, provided $0 < \alpha' < \alpha$ and $0 < \beta_l' < \beta_l$ for all l such that $g_l \neq 0$. Similarly, $B^{\alpha,\overline{\beta}}(\omega) \in C^{\alpha'}([0, T], H_q^{\beta'}(\mathbb{R}^n, \mathbb{R}))$, if $0 < \alpha' < \alpha$ and $0 < \beta < \beta_l$ for all l.*

(ii) *The hybrid fractional Brownian sheet $B^{\alpha,\beta}$ possesses a modification, again denoted by $B^{\alpha,\beta}$, such that (with the above convention) for $\mathbb{P}$-a.e. $\omega \in \Omega$, $B^{\alpha,\beta}(\omega) \in C^{\alpha'}([0, T], H_q^{\beta'}(\mathbb{R}^n, \mathbb{R}))$, provided $0 < \alpha' < \alpha$ and $0 < \beta' < \beta$.*

This allows to indicate some applications of the discussed parabolic models with z respectively the Z^j's being paths of fractional Brownian sheets. Then also the equations are to be read in the pathwise sense: There is some $\Omega_1 \subset \Omega$, $\mathbb{P}(\Omega_1) = 1$, such that for any $\omega \in \Omega_1$, solutions to (4.5), (4.14) and (4.17) are obtained for $Z(\omega)$ in place of Z. Note that restricted to Ω_1, the pathwise estimates in the corresponding proofs remain valid.

Statistical coupling. Assume that $Z = (B^{\alpha,\beta}, ..., B^{\alpha,\beta})$ is a k-vector with the same single hybrid fractional Brownian sheet in each component. If the cross-diffusion matrix B is diagonal, the linear problem (4.17) models a decoupled system under a common random force. With general B, the same system is coupled. In this case, the (4.5) and (4.14) describe coupled systems with a common random potential.

Coupled systems and mixed noises. Assume $Z = (B_1^{\alpha^1,\beta^1}, ..., B_k^{\alpha^k,\beta^k})$ is an independent vector of hybrid fractional sheets $B_j^{\alpha^j,\beta^j}$ of (possibly different) orders $0 < \alpha^j, \beta^j < 1$. Then (4.5) and (4.14) yield systems coupled by cross-diffusion, non-linearity, or by a (mixed) fractional noise term. If $\min_j \alpha^j$ and $\min_j \beta^j$ are sufficiently large, we obtain function solutions.
The case $\alpha^j = \alpha$, $\beta^j = \beta$, $j = 1, ..., k$, that is $Z = (B_1^{\alpha,\beta}, ..., B_k^{\alpha,\beta})$, provides the simplest prototype.

Anisotropic noises. Let $Z = (B_1^{\alpha,\overline{\beta^1}}, ..., B_k^{\alpha,\overline{\beta^k}})$ be an independent vector consisting of of anisotropic fractional Brownian sheets $B_j^{\alpha,\overline{\beta^j}}$, $0 < \alpha < 1$, $\overline{\beta^j} = (\beta_1^j, ..., \beta_n^j)$, $0 < \beta_l^j < 1$. With the refinements described in Corollary 4.4.1, we may obtain function solutions to (4.5), (4.14) or (4.17), as long as $\beta < \beta_l^j$ for all j, l for which G_l^j does not vanish identically.

Remark 4.6.1.

(i) In the special case $k = n = 1$, Theorem 4.3.1 yields both *existence and uniqueness* of solutions to a class of *one-dimensional semilinear heat equations driven by anisotropic fractional Brownian sheets* $B^{1-\alpha,1-\beta}$:

$$\frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + F(u(t,x)) + G(u(t,x))\frac{\partial^2 B^{1-\alpha,1-\beta}}{\partial t \partial x}(t,x) \ . \quad (4.24)$$

This is closely related to results in [28], [36], [43] and [44].

(ii) If the noise is white in space, Theorem 4.3.1 requires roughly speaking $2\alpha < 2\gamma < 1 - 2\alpha$. We end up with the necessary condition $\alpha < 1/4$. In other words, *the temporal regularity* $1 - \alpha$ *of the driving* (temporal Hurst index in the case of fractional Brownian sheets) needs to be greater than $3/4$ in order to get function solutions. This threshold has also been observed in specific examples in [56], [36] and [44].

(iii) For $k = n = 1$, the linear model (4.17) leads to mild solutions u whose temporal and spatial regularity behaviour γ and δ are in good accordance with the classical results of [84]. See Proposition 3.7 there and consider (roughly speaking) $\alpha = \beta = 1/2$ in our Theorem 4.5.1.

(iv) In the model with additive noise, we may of course choose space-time white noise.

(v) We wish to point out that the (components of the) driving field Z do(es) not necessarily need to be of fractional Brownian type, the approach relies only on Hölder properties. For instance, one may consider as well fractional stable sheets with suitable parameters.

Chapter 5

Proofs

The existence and uniqueness statements of the preceding sections rely on the mapping properties of the integral operator, which are investigated in the following. As a by-product we deduce the correctness of Definition 4.1.1, interpreted according to the respective situations of Theorems 4.2.1, 4.3.1 and 4.5.1. Finally, we conclude the main statements of Chapter 4.

5.1 Mapping properties

Recall the definitions (4.9), (4.10) and (4.15) of the spaces $C^\gamma([0,T], H_q^\delta(\mathbb{R}^n,\mathbb{R}^k))$, $W^\gamma([0,T], \mathring{H}_2^\delta(D,\mathbb{R}^k))$ and $W^\gamma([0,T], \mathring{H}_{2,\infty}^\delta(D,\mathbb{R}^k))$.
The main steps in proving Theorems 4.2.1, 4.3.1 and 4.5.1 are formulated in the following three propositions:

Proposition 5.1.1. *Given $G = (G_l^j) \in \mathcal{M}(n\times k, \mathbb{R})$ and $0 < \alpha, \beta, \gamma < 1$, $\delta > 0$, the mapping*

$$Z \mapsto I_{(\cdot)}\left(G, \frac{\partial}{\partial t}\nabla Z\right) \tag{5.1}$$

is a continuous linear operator from $C^{1-\alpha}([0,T], H_2^{1-\beta}(\mathbb{R}^n,\mathbb{R}^k))$ into $C^\gamma([0,T], H_2^\delta(D,\mathbb{R}^k))$, provided $\gamma+\alpha < 1$ and $2\gamma+\delta < 2-2\alpha-\beta$.

We introduce the following equivalent norms on $W^\gamma([0,T], \mathring{H}_2^\delta(D,\mathbb{R}^k))$:

$$\|u\|_{\gamma,\delta}^{(\varrho)} := \sup_{0\le t\le T} e^{-\varrho t}\left(\|u(t)\|_\delta + \int_0^t \frac{\|u(t)-u(\tau)\|_\delta}{(t-\tau)^{\gamma+1}} d\tau\right) < \infty, \tag{5.2}$$

where $\varrho \geq 1$ is a parameter, cf. [56].

Proposition 5.1.2. *Let $0 < \alpha, \beta, \gamma < 1$, $\alpha < \gamma < 1-\alpha$, $\beta < \delta$ and $2\gamma + \delta < 2 - 2\alpha - \beta$.*
For $Z \in C^{1-\alpha}([0,T], H_q^{1-\beta}(\mathbb{R}^n, \mathbb{R}^k))$, $q > 2 \vee (n/\delta)$, and $G = (G_l^j)$ such that each G_l is linear, the mapping

$$u \mapsto I_{(\cdot)}\left(G(u), \frac{\partial}{\partial t}\nabla Z\right) \tag{5.3}$$

is a contraction in $W^\gamma([0,T], \mathring{H}_2^\delta(D, \mathbb{R}^k))$. More precisely,

$$\left\| I_{(\cdot)}\left(G(u), \frac{\partial}{\partial t}\nabla Z\right)\right\|_{\gamma,\delta}^{(\varrho)} \leq C(\varrho)\ \|u\|_{\gamma,\delta}^{(\varrho)}\ ,$$

where $C(\varrho) > 0$ tends to zero as ϱ goes to infinity.

Now let the equivalent norms $\|\cdot\|_{\gamma,\delta,\infty}^{(\varrho)}$ in $W^\gamma([0,T], \mathring{H}_{2,\infty}^\delta(D, \mathbb{R}^k))$ be defined as the analogues of (5.2), based on (4.15) i.e.

$$\|u\|_{\gamma,\delta,\infty}^{(\varrho)} := \sup_{0 \leq t \leq T}\left(\|u(t)\|_{\delta,\infty} + \int_0^t \frac{\|u(t) - u(\tau)\|_{\delta,\infty}}{(t-\tau)^{\gamma+1}} d\tau\right)$$

.

Proposition 5.1.3. *Let $0 < \alpha, \beta, \gamma, \delta < 1$. Further assume that $\alpha < \gamma < 1 - \alpha$, $\beta < \delta$ and $2\gamma + \delta \vee (n/2) < 2 - 2\alpha - \beta$.*
Let $Z \in C^{1-\alpha}([0,T], H_q^{1-\beta}(\mathbb{R}^n, \mathbb{R}^k))$, $q > 2 \vee (n/\delta)$, and let the non-linear coefficient term $G = (G_1, ..., G_n)$, be such that $G_l(0) = 0$, and each G_l has bounded and Lipschitz second order differential $\mathcal{D}^2 G$, i.e. (4.12) and (4.13) hold.
Then there is a closed ball $B_0 \subset W^\gamma([0,T], \mathring{H}_{2,\infty}^\delta(D, \mathbb{R}^k))$, such that (5.3) maps B_0 into itself and for $\varrho \geq 1$ large enough,

$$\left\| I_{(\cdot)}\left(G(u), \frac{\partial}{\partial t}\nabla Z\right) - I_{(\cdot)}\left(G(v), \frac{\partial}{\partial t}\nabla Z\right)\right\|_{\gamma,\delta,\infty}^{(\varrho)} \leq C(\varrho)\ \|u - v\|_{\gamma,\delta,\infty}^{(\varrho)}\ , \tag{5.4}$$

$u, v \in B_0$.

Below we give a detailed proof of Proposition 5.1.2. Then we discuss some modifications that establish also the proofs of Propositions 5.1.1 and 5.1.3. The following remarks are useful.

Remark 5.1.1.

(i) We point out that for $\gamma' > \gamma$, $W^{\gamma'}([0,T], \mathring{H}_2^\delta(D,\mathbb{R}^k))$ with norm $\|\cdot\|^{(\varrho)}_{\gamma',\delta}$ *is continuously embedded into* $W^{\gamma}([0,T], \mathring{H}_2^\delta(D,\mathbb{R}^k))$ with norm $\|u\|^{(\varrho)}_{\gamma,\delta}$ (with the same parameter ϱ).

(ii) Note that in Definition 4.1.1 and according to (2.2), we have $I_t^\alpha\left(g, \frac{\partial}{\partial t}\nabla Z\right) := \sum_{l=1}^n I_t^{(l)}(g_l \cdot \frac{\partial^2 Z}{\partial t \partial x_l})$, $t>0$, where

$$I_t^{(l)}(g_l \cdot \frac{\partial^2 Z}{\partial t\partial x_l}) := (-1)^\alpha \int_0^t A^\alpha P(t-s)\left(g_l(s)\cdot \frac{\partial}{\partial x_l} D_{t-}^{1-\alpha} Z_t(s)\right) ds \tag{5.5}$$

$$+ c_\alpha (-1)^\alpha \int_0^t \int_0^s (s-\sigma)^{-\alpha-1} P(t-\sigma)\left((g_l(s)-g_l(\sigma))\cdot \frac{\partial}{\partial x_l} D_{t-}^{1-\alpha} Z_t(s)\right) d\sigma ds$$

$$+ c_\alpha (-1)^\alpha \int_0^t \int_s^\infty \sigma^{-\alpha-1} P(\sigma+t-s)\left(g_l(s)\cdot \frac{\partial}{\partial x_l} D_{t-}^{1-\alpha} Z_t(s)\right) d\sigma ds\ .$$

We verfiy Proposition 5.1.2.

Proof. Step 1: Parameters. We write α' to denote the number α as given in Proposition 5.1.2, i.e. by hypothesis, $Z \in C^{1-\alpha'}([0,T], H_q^{1-\beta}(\mathbb{R}^n,\mathbb{R}^k))$, $\alpha' < \gamma < 1-\alpha'$ and $2\gamma + 2\alpha' + \delta + \beta < 2$. Consequently there exists some small $\mu > 0$ such that with $\alpha := \alpha' + \mu$, we still have $\alpha < \gamma < 1-\alpha$ and

$$2\gamma + 2\alpha + \delta + \beta < 2\ . \tag{5.6}$$

In Definition 4.1.1, we use the number α as specified this way.
For later use, we record the relation

$$\int_0^t e^{-\varrho(t-s)} s^{-\eta}(t-s)^{-\theta} ds \le \varrho^{\eta+\theta-1}\left(\sup_{z>0}\int_0^z e^{-v}(z-v)^{-\eta} v^{-\theta} dv\right)\ , \tag{5.7}$$

$0 < \eta, \theta < \eta+\theta < 1$, the supremum is bounded by $2 + 4B(1-\eta, 1-\theta)$, B denoting the Beta function.

Step 2: Estimates on the correction terms. Recall that

$$D_{t-}^{1-\alpha} Z_t(s) = \mathbf{1}_{(0,t)}(s)\frac{(-1)^\alpha}{\Gamma(1-\alpha)}\left(\frac{Z(s)-Z(t)}{(t-s)^{1-\alpha}} + \alpha\int_s^t \frac{Z(s)-Z(\sigma)}{(\sigma-s)^{2-\alpha}}\, d\sigma\right)\ . \tag{5.8}$$

The first term in brackets (boundary correction term) will be denoted by $b(s,t)$, the second (integral term) by $j(s,t)$. Recall the definition (4.9) of the Hölder norm $|\cdot|_{1-\alpha',1-\beta}$. Note that with q as specified and $0 \le s < t \le T$,

$$\left\|b(s,t)|H_q^{1-\beta}(\mathbb{R}^n,\mathbb{R}^k)\right\| \le c(t-s)^\mu \left\|Z|C^{1-\alpha'}([0,T],H_q^{1-\beta}(\mathbb{R}^n,\mathbb{R}^k))\right\|$$
$$\text{and} \quad \left\|j(s,t)|H_q^{1-\beta}(\mathbb{R}^n,\mathbb{R}^k)\right\| \le c(t-s)^\mu \left\|Z|C^{1-\alpha'}([0,T],H_q^{1-\beta}(\mathbb{R}^n,\mathbb{R}^k))\right\| . \tag{5.9}$$

In particular, $\left\|D_{t-}^{1-\alpha}Z_t(s)|H_q^{1-\beta}(\mathbb{R}^n,\mathbb{R}^k)\right\| < c$. As Z is fixed throughout the whole proof, we absorb the Hölder norm $\left\|Z|C^{1-\alpha'}([0,T],H_q^{1-\beta}(\mathbb{R}^n,\mathbb{R}^k))\right\|$ of Z into the constants c to simplify the notation.
For $0 \le s < \tau < t \le T$ one deduces

$$\begin{aligned}&\left\|b(s,t)-b(s,\tau)|H_q^{1-\beta}(\mathbb{R}^n,\mathbb{R}^k)\right\| \\ &\qquad \le c(t-s)^\mu(\tau-s)^{\alpha-1}(t-\tau)^{1-\alpha} + c(t-\tau)^{1-\alpha+\mu}(\tau-s)^{\alpha-1} ,\end{aligned} \tag{5.10}$$

or, alternatively,

$$\begin{aligned}&\left\|b(s,t)-b(s,\tau)|H_q^{1-\beta}(\mathbb{R}^n,\mathbb{R}^k)\right\| \\ &\qquad \le c(t-\tau)^{1-\alpha+\mu}(\tau-s)^{1-\alpha} + c(t-\tau)^{1-\alpha}(\tau-s)^{1-\alpha+\mu} .\end{aligned} \tag{5.11}$$

The constants c may depend on q.

Step 3: The non-difference part. Recall the definition (5.2) of the norms $\|\cdot\|_{\gamma,\delta}^{(\varrho)}$. We begin with an estimate on the first term in brackets there. Fix $l = 1,...,n$ and denote by $J_1(t)$, $J_2(t)$ and $J_3(t)$ the summands according to the right hand side of (5.5) in the order they occur. We consider $G_l(u(s))$ in place of $g_l(s)$ and write G to abbreviate G_l.
By Lemma 2.3.1 and a simple Fourier multiplier argument, we have

$$\begin{aligned}\left\|G(u(s))\cdot\frac{\partial}{\partial x_l}D_{t-}^{1-\alpha}Z_t(s)\right\|_{-\beta} &\le c\,\|G(u(s))\|_\delta \left\|\frac{\partial}{\partial x_l}D_{t-}^{1-\alpha}Z_t(s)|H_q^{-\beta}(\mathbb{R}^n,\mathbb{R}^k)\right\| \\ &\le c\,\|u(s)\|_\delta \left\|b(s,t)+j(s,t)|H_q^{1-\beta}(\mathbb{R}^n,\mathbb{R}^k)\right\|\end{aligned} \tag{5.12}$$

for some $q > 1$. Recall that $\|\cdot\|_\delta$ is our abbreviation for the norm $\left\|\cdot|H_2^\delta(\mathbb{R}^n,\mathbb{R}^k)\right\|$.
Now set

$$\kappa := \frac{\delta+\beta}{2}$$

and use the mapping property (2.11) of $(P(t))_{t\geq 0}$ together with (5.9) and (5.12) to obtain

$$
\begin{aligned}
e^{-\varrho t}\,\|J_1(t)\|_\delta &\leq c\,e^{-\varrho t}\int_0^t (t-s)^{-\alpha-\kappa}\,\|u(s)\|_\delta\,ds \\
&\leq c\,\|u\|_{\gamma,\delta}^{(\varrho)}\int_0^t e^{-\varrho(t-s)}(t-s)^{-\alpha-\kappa}ds \\
&\leq c\,\|u\|_{\gamma,\delta}^{(\varrho)}\,\varrho^{\alpha+\kappa-1}\ .
\end{aligned}
$$

Similarly, by (5.9),

$$
\begin{aligned}
e^{-\varrho t}\,\|J_2(t)\|_\delta &\leq c\,e^{-\varrho t}\int_0^t\int_0^s (t-\sigma)^{-\kappa}\frac{\|u(s)-u(\sigma)\|_\delta}{(s-\sigma)^{\alpha+1}}\,d\sigma ds \\
&\leq c\,\|u\|_{\gamma,\delta}^{(\varrho)}\,\varrho^{\kappa-1}\ ,
\end{aligned}
$$

recall $\alpha<\gamma$ and the remark preceding this proof. Note also that $(t-s)<(t-\sigma)$ and $0<\kappa<1$.
Finally, due to (5.7),

$$
\begin{aligned}
e^{-\varrho t}\,\|J_3(t)\|_\delta &\leq c\,e^{-\varrho t}\int_0^t s^{-\alpha}(t-s)^{-\kappa}\,\|u(s)\|_\delta\ ds \\
&\leq c\,\|u\|_{\gamma,\delta}^{(\varrho)}\,\varrho^{\alpha+\kappa-1}\ .
\end{aligned}
$$

Consequently,

$$
e^{-\varrho t}\,\|J_i(t)\|_\delta \leq C_0(\varrho)\,\|u\|_{\gamma,\delta}^{(\varrho)}\quad ,\quad i=1,2,3
$$

for any $0\leq t\leq T$ and with $C_0(\varrho)>0$ tending to zero as ϱ goes to infinity.

Step 4: The difference part and J_1. Turning to estimates on the difference part of the norms (5.2), we start with J_1. For $0 \le \tau < t \le T$,

$$\begin{aligned}
& c(J_1(t) - J_1(\tau)) \\
&= \int_0^t A^\alpha P(t-s)G(u(s))\cdot\frac{\partial}{\partial y_l}D_{t-}^{1-\alpha}Z_t(s)ds - \int_0^\tau A^\alpha P(\tau-s)G(u(s))\cdot\frac{\partial}{\partial y_l}D_{\tau-}^{1-\alpha}Z_\tau(s)ds \\
&= \int_0^t A^\alpha P(t-s)G(u(s))\cdot\frac{\partial}{\partial y_l}D_{t-}^{1-\alpha}Z_t(s)ds - \int_0^\tau A^\alpha P(t-s)G(u(s))\cdot\frac{\partial}{\partial y_l}D_{\tau-}^{1-\alpha}Z_\tau(s)ds \\
&+ \int_0^\tau A^\alpha P(t-s)G(u(s))\cdot\frac{\partial}{\partial y_l}D_{\tau-}^{1-\alpha}Z_\tau(s)ds - \int_0^\tau A^\alpha P(\tau-s)G(u(s))\cdot\frac{\partial}{\partial y_l}D_{\tau-}^{1-\alpha}Z_\tau(s)ds \\
&= \int_\tau^t A^\alpha P(t-s)G(u(s))\cdot\frac{\partial}{\partial y_l}D_{t-}^{1-\alpha}Z_t(s)ds \quad - \int_0^\tau A^\alpha P(t-s)G(u(s))\cdot \\
&\qquad \cdot\frac{\partial}{\partial y_l}(D_{t-}^{1-\alpha}Z_t(s) - D_{\tau-}^{1-\alpha}Z_\tau(s))ds \\
&\qquad + \int_0^\tau A^\alpha[P(t-\tau) - I]P(\tau-s)G(u(s))\cdot\frac{\partial}{\partial y_l}D_{\tau-}^{1-\alpha}Z_\tau(s)ds\ , \qquad (5.13)
\end{aligned}$$

we have used the semigroup property of $(P(t))_{t\ge0}$.
By the mapping properties (2.11) and (2.12) of $(P(t))_{t\ge0}$, the $\|\cdot\|_\delta$-norm of the last term on the right hand side of (5.13) admits the bound

$$c(t-\tau)^\nu \int_0^\tau (t-s)^{-\alpha-\kappa-\nu}\left\|u(s)\right\|_\delta\ ds \qquad (5.14)$$

with some $\gamma < \nu < 1$ being just slightly bigger than γ. Integrating against $(t-\tau)^{-\gamma-1}d\tau$ over $(0,t)$, and multiplying by $e^{-\varrho t}$, we are led to the bound

$$\begin{aligned}
& c\left\|u\right\|_{\gamma,\delta}^{(\varrho)}\int_0^t\int_0^\tau e^{-\varrho(t-s)}(t-s)^{-\alpha-\kappa-\nu}ds(t-\tau)^{\nu-\gamma-1}d\tau \\
&\qquad \le c\left\|u\right\|_{\gamma,\delta}^{(\varrho)}\int_0^t\int_0^t e^{-\varrho(t-s)}(t-s)^{-\alpha-\kappa-\nu}ds(t-\tau)^{\nu-\gamma-1}d\tau \\
&\qquad\qquad \le c\left\|u\right\|_{\gamma,\delta}^{(\varrho)}\varrho^{\alpha+\kappa+\nu-1}\ .
\end{aligned}$$

For the middle summand on the right hand side of (5.13), consider

$$D_{t-}^{1-\alpha}Z_t(s) - D_{\tau-}^{1-\alpha}Z_\tau(s) = c(b(s,t) - b(s,\tau)) + c(j(s,t) - j(s,\tau))\ .$$

With $c(b(s,t)-b(s,\tau))$ in place of $D_{t-}^{1-\alpha}Z_t(s) - D_{\tau-}^{1-\alpha}Z_\tau(s)$ in that summand, (5.10) yields

$$c\int_0^\tau (t-s)^{-\alpha-\kappa}\left\|u(s)\right\|_\delta(\tau-s)^{\alpha-1}(t-\tau)^{1-\alpha}(t-s)^\mu ds \qquad (5.15)$$

plus

$$c\int_0^\tau (t-s)^{-\alpha-\kappa}\,\|u(s)\|_\delta\,(\tau-s)^{\alpha-1}(t-\tau)^{1-\alpha+\mu}ds\ , \tag{5.16}$$

and after integration,

$$\begin{aligned}
&c\,\|u\|_{\gamma,\delta}^{(\varrho)}\int_0^t\int_0^\tau e^{-\varrho(t-s)}(t-s)^{\mu-\alpha-\kappa}(\tau-s)^{\alpha-1}ds(t-\tau)^{-\alpha-\gamma}d\tau\\
&\quad\le c\,\|u\|_{\gamma,\delta}^{(\varrho)}\,B(\alpha,1-\gamma-\alpha)\int_0^t e^{-\varrho(t-s)}(t-s)^{\mu-\gamma-\alpha-\kappa}ds\\
&\quad\le c\,\|u\|_{\gamma,\delta}^{(\varrho)}\,\varrho^{\alpha+\gamma+\kappa-\mu-1}\ ,
\end{aligned}$$

plus

$$\begin{aligned}
c\,\|u\|_{\gamma,\delta}^{(\varrho)}\int_0^t\int_0^\tau e^{-\varrho(t-s)}(t-s)^{-\alpha-\kappa}(\tau-s)^{\alpha-1}ds(t-\tau)^{\mu-\alpha-\gamma}d\tau&\\
\le c\,\|u\|_{\gamma,\delta}^{(\varrho)}\,\varrho^{\alpha+\gamma+\kappa-\mu-1}&\ ,
\end{aligned}$$

which follows by similar arguments. Recall that $\gamma<1-\alpha$.
For the same summand with

$$j(s,t)-j(s,\tau)=c\int_\tau^t\frac{Z(s)-Z(\sigma)}{(\sigma-s)^{2-\alpha}}d\sigma$$

in place of $D_{t-}^{1-\alpha}Z_t(s)-D_{\tau-}^{1-\alpha}Z_\tau(s)$, we use Fubini's theorem to observe that the $\|\cdot\|_\delta$-norm of

$$c\int_0^\tau A^\alpha P(t-s)G(u(s))\cdot\frac{\partial}{\partial y_l}(j(s,t)-j(s,\tau))ds$$

is bounded above by

$$\begin{aligned}
&c\int_0^\tau\int_\tau^t(t-s)^{-\kappa-\alpha}\,\|u(s)\|_\delta\,(\sigma-s)^{\mu-1}d\sigma ds\\
&\qquad\le c\int_\tau^t\int_0^\tau(\sigma-s)^{\mu-\alpha-\kappa-1}\,\|u(s)\|_\delta\,dsd\sigma\\
&\qquad\le c\,\|u\|_{\gamma,\delta}^{(\varrho)}\int_\tau^t\int_0^\tau e^{\varrho s}(\sigma-s)^{\mu-\alpha-\kappa-1}dsd\sigma\\
&\qquad\le c\,\|u\|_{\gamma,\delta}^{(\varrho)}\,e^{\varrho\tau}\int_\tau^t(\sigma^{\mu-\alpha-\kappa}+(\sigma-\tau)^{\mu-\alpha-\kappa})d\sigma\\
&\qquad\qquad\le ce^{\varrho\tau}(t-\tau)^{1-\alpha-\kappa+\mu}\,\|u\|_{\gamma,\delta}^{(\varrho)},
\end{aligned}\tag{5.17}$$

note that $s < \tau < \sigma < t$ and $0 < \alpha + \kappa - \mu < 1$. Integrating with respect to $(t-\tau)^{-\gamma-1}d\tau$ and taking into account the exponential factors, we obtain the estimate

$$c\,\|u\|^{(\varrho)}_{\gamma,\delta} \int_0^t e^{-\varrho(t-\tau)}(t-\tau)^{-\alpha-\gamma-\kappa+\mu}d\tau \le c\,\|u\|^{(\varrho)}_{\gamma,\delta}\,\varrho^{\alpha+\gamma+\kappa-\mu-1}\ .$$

Turn to the first summand on the right hand side of (5.13). In the $\|\cdot\|_\delta$-norm it is bounded above by

$$c(t-\tau)^{\nu}\int_\tau^t (t-s)^{-\alpha-\kappa-\nu}\,\|u(s)\|_\delta\,ds\ , \tag{5.18}$$

ν again just sligtly bigger than γ. Integration leads to the bound

$$\begin{aligned}
&c\,\|u\|^{(\varrho)}_{\gamma,\delta}\int_0^t\int_\tau^t e^{-\varrho(t-s)}(t-s)^{-\alpha-\kappa-\nu}ds(t-\tau)^{\nu-\gamma-1}d\tau \\
&\qquad\le c\,\|u\|^{(\varrho)}_{\gamma,\delta}\int_0^t e^{-\varrho(t-\tau)}\int_\tau^t (t-s)^{-\alpha-\kappa-\nu}ds(t-\tau)^{\nu-\gamma-1}d\tau \\
&\qquad\qquad\le c\,\|u\|^{(\varrho)}_{\gamma,\delta}\,\varrho^{\alpha+\gamma+\kappa-1}\ .
\end{aligned}$$

Clipping the estimates, we see that for $0 \le t \le T$,

$$e^{-\varrho t}\int_0^t \frac{\|J_1(t)-J_1(\tau)\|_\delta}{(t-\tau)^{\gamma+1}}d\tau \le C_1(\varrho)\,\|u\|^{(\varrho)}_{\gamma,\delta}\ ,$$

$C_1(\varrho)$ tending to zero as ϱ goes to infinity.

Step 5: The difference part and J_2. For $0 \le \tau < t \le T$, we split the differences of J_2 similarly to those of J_1:

$$
\begin{aligned}
&c(J_2(t)-J_2(\tau)) = \\
&= \int_0^t \int_0^s (s-\sigma)^{-\alpha-1} P(t-\sigma)(G(u(s))-G(u(\sigma))) \cdot \frac{\partial}{\partial y_l} D_{t-}^{1-\alpha} Z_t(s)\, d\sigma ds \\
&- \int_0^\tau \int_0^s (s-\sigma)^{-\alpha-1} P(\tau-\sigma)(G(u(s))-G(u(\sigma))) \cdot \frac{\partial}{\partial y_l} D_{\tau-}^{1-\alpha} Z_\tau(s)\, d\sigma ds \\
&= \int_\tau^t \int_0^s (s-\sigma)^{-\alpha-1} P(t-\sigma)(G(u(s))-G(u(\sigma))) \cdot \frac{\partial}{\partial y_l} D_{t-}^{1-\alpha} Z_t(s)\, d\sigma ds \\
&\qquad - \int_0^\tau \int_0^s (s-\sigma)^{-\alpha-1} P(t-\sigma)(G(u(s))-G(u(\sigma))) \cdot \\
&\qquad\qquad \cdot \frac{\partial}{\partial y_l} (D_{t-}^{1-\alpha} Z_t(s) - D_{\tau-}^{1-\alpha} Z_\tau(s))\, d\sigma ds \\
&\quad + \int_0^\tau \int_0^s (s-\sigma)^{-\alpha-1} [P(t-\tau)-I] P(\tau-\sigma)(G(u(s))-G(u(\sigma))) \cdot \\
&\qquad\qquad\qquad \cdot \frac{\partial}{\partial y_l} D_{\tau-}^{1-\alpha} Z_\tau(s)\, d\sigma ds\ . \qquad (5.19)
\end{aligned}
$$

Since $P(t-\sigma) = P(t-s)P(s-\sigma)$, $0<\sigma<s<t$, the first summand after the last equality sign admits the bound

$$\int_\tau^t (t-s)^{-\kappa} \int_0^s \frac{\|u(s)-u(\sigma)\|_\delta}{(s-\sigma)^{\alpha+1}} d\sigma ds\ ,$$

using $t-s<t-\tau$ and integrating, we arrive at

$$c\,\|u\|_{\gamma,\delta}^{(\varrho)} \int_0^t e^{-\varrho(t-s)} (t-s)^{-\kappa-\nu} \int_0^s (t-\tau)^{\nu-\gamma-1} d\tau ds \le c\,\|u\|_{\gamma,\delta}^{(\varrho)}\, \varrho^{\kappa+\nu-1},$$

where again ν is chosen slightly bigger than γ.
An estimate of the last summand in (5.19) follows as before combining (2.11) and (2.12), its norm contributes at most

$$c \int_0^\tau (t-\tau)^\nu (\tau-s)^{-\kappa-\nu} \int_0^s \frac{\|u(s)-u(\sigma)\|_\delta}{(s-\sigma)^{\alpha+1}} d\sigma ds\ ,$$

and after integration, it remains less or equal

$$c\,\|u\|_{\gamma,\delta}^{(\varrho)}\, \varrho^{\kappa+\nu-1}\ .$$

Now consider the middle summand on the right hand side of the last equality in (5.19) with $c(b(s,t)-b(s,\tau))$ in place of $D^{1-\alpha}_{t-}Z_t(s)-D^{1-\alpha}_{\tau-}Z_\tau(s)$. Using (5.11) we observe the bound

$$c\int_0^\tau (\tau-s)^{-\kappa}\int_0^s \frac{\|u(s)-u(\sigma)\|_\delta}{(s-\sigma)^{\alpha+1}}d\sigma\times \\ \times\left((t-\tau)^{1-\alpha+\mu}(\tau-s)^{1-\alpha}+(t-\tau)^{1-\alpha}(\tau-s)^{1-\alpha+\mu}\right)ds\ .$$

Integration yields

$$c\,\|u\|^{(\varrho)}_{\gamma,\delta}\int_0^t\int_0^\tau e^{-\varrho(t-s)}\left((\tau-s)^{1-\alpha-\kappa}(t-\tau)^{-\alpha-\gamma+\mu}+(\tau-s)^{1-\alpha-\kappa+\mu}(t-\tau)^{-\alpha-\gamma}\right)dsd\tau \\ \le c\,\|u\|^{(\varrho)}_{\gamma,\delta}\,\varrho^{2\alpha+\gamma+\kappa-\mu-2}\ ,$$

note that $\alpha+\gamma<1$. We have used (5.6) to see that

$$\int_0^t e^{-\varrho(t-\tau)}\tau^{2-\alpha-\kappa}(t-\tau)^{-\alpha-\gamma+\mu}d\tau\le c\varrho^{2\alpha+\kappa+\gamma-\mu-2}\ ,$$

and similarly for the other term. For the same summand with $j(s,t)-j(s,\tau)$ inserted, Fubini's theorem again tells that the norm does not exceed

$$\int_0^\tau\int_\tau^t (t-s)^{-\kappa}\int_0^s\frac{\|u(s)-u(\sigma)\|_\delta}{(s-\sigma)^{\alpha+1}}\,d\sigma(\theta-s)^{\mu-1}\,d\theta ds \\ \le c\,\|u\|^{(\varrho)}_{\gamma,\delta}\,e^{\varrho\tau}\int_\tau^t\int_0^\tau(\theta-s)^{\mu-\kappa-1}dsd\theta \\ \le c\,\|u\|^{(\varrho)}_{\gamma,\delta}\,e^{\varrho\tau}\int_\tau^t(\theta^{\mu-\kappa}+(\theta-\tau)^{\mu-\kappa})d\theta \\ \le c\,\|u\|^{(\varrho)}_{\gamma,\delta}\,e^{\varrho\tau}(t-\tau)^{1+\mu-\kappa}\ .$$

Performing the integration with respect to τ, we obtain no more than

$$c\,\|u\|^{(\varrho)}_{\gamma,\delta}\int_0^t e^{-\varrho(t-\tau)}(t-\tau)^{-\gamma-\kappa+\mu}d\tau\le c\,\|u\|^{(\varrho)}_{\gamma,\delta}\,\varrho^{\gamma+\kappa-\mu-1}\ .$$

Combining these estimates, we see that

$$e^{-\varrho t}\int_0^t\frac{\|J_2(t)-J_2(\tau)\|_\delta}{(t-\tau)^{\gamma+1}}d\tau\le C_2(\varrho)\,\|u\|^{(\varrho)}_{\gamma,\delta}\,,$$

$C_2(\varrho)$ tending to zero as ϱ goes to infinity.

Step 6: The difference part and J_3. Splitting the difference as before,

$$
\begin{aligned}
&c(J_3(t) - J_3(\tau)) = \\
&\qquad \int_\tau^t \int_s^\infty \sigma^{-\alpha-1} P(\sigma) P(t-s) G(u(s)) \cdot \frac{\partial}{\partial y_l} D_{t-}^{1-\alpha} Z_t(s)\, d\sigma ds \\
&+ \int_0^\tau \int_s^\infty \sigma^{-\alpha-1} P(\sigma)[P(t-\tau) - I] P(\tau - s) G(u(s)) \cdot \frac{\partial}{\partial y_l} D_{\tau-}^{1-\alpha} Z_\tau(s)\, d\sigma ds \\
&- \int_0^\tau \int_s^\infty \sigma^{-\alpha-1} P(\sigma) P(t-s) G(u(s)) \cdot \frac{\partial}{\partial y_l} (D_{t-}^{1-\alpha} Z_t(s) - D_{\tau-}^{1-\alpha} Z_\tau(s)\, d\sigma ds\ .
\end{aligned}
\tag{5.20}
$$

The norm of the first term on the right hand side does not exceed

$$
c \int_\tau^t s^{-\alpha} (t-s)^{-\kappa} \|u(s)\|_\delta\, ds\ . \tag{5.21}
$$

Since here $t - s < t - \tau$, integration yields

$$
c \|u\|_{\gamma,\delta}^{(\varrho)} \int_0^t \int_0^t e^{-\varrho(t-s)} s^{-\alpha} (t-s)^{-\kappa-\nu} ds (t-\tau)^{\nu-\gamma-1} d\tau \le c \|u\|_{\gamma,\delta}^{(\varrho)} \varrho^{\alpha+\kappa+\nu-1}
$$

with some ν slightly bigger than γ, in particular $\alpha + \kappa + \nu < 1$. We have used (5.7).
The second summand in (5.20) contributes

$$
\int_0^\tau s^{-\alpha} (t-\tau)^\nu (\tau - s)^{-\kappa-\nu} \|u(s)\|_\delta\, ds \tag{5.22}
$$

with some $\nu > \gamma$, but close. Integrating and sorting out a Beta function, we arrive at

$$
\begin{aligned}
c \|u\|_{\gamma,\delta}^{(\varrho)} &\int_0^t e^{-\varrho(t-\tau)} \int_0^\tau s^{-\alpha} (\tau - s)^{-\kappa-\nu} ds (t-\tau)^{\nu-\gamma-1} d\tau \\
&\le c \|u\|_{\gamma,\delta}^{(\varrho)} \int_0^t e^{-\varrho(t-\tau)} \tau^{1-\alpha-\kappa-\nu} (t-\tau)^{\nu-\gamma-1} d\tau \\
&\qquad\qquad \le c \|u\|_{\gamma,\delta}^{(\varrho)} \varrho^{\alpha+\gamma+\kappa-1}
\end{aligned}
$$

by (5.7).

The third term in (5.20) with the boundary terms $c(b(s,t)-b(s,\tau))$ in place of $D^{1-\alpha}_{t-}Z_t - D^{1-\alpha}_{\tau-}Z_\tau(s)$ contributes

$$c\int_0^\tau s^{-\alpha}(\tau-s)^{-\kappa}\left\|u(s)\right\|_\delta((t-\tau)^{1-\alpha+\mu}(\tau-s)^{1-\alpha}+(\tau-s)^{1-\alpha+\mu}(t-\tau)^{1-\alpha})dsds\,, \tag{5.23}$$

here we have used (5.11). For the first summand, integration and evaluation of a Beta function yield

$$\begin{aligned}c\left\|u\right\|^{(\varrho)}_{\gamma,\delta}&\int_0^t\int_0^\tau e^{-\varrho(t-s)}s^{-\alpha}(\tau-s)^{1-\alpha-\kappa}ds(t-\tau)^{-\gamma-\alpha+\mu}d\tau\\&\le c\left\|u\right\|^{(\varrho)}_{\gamma,\delta}\int_0^t e^{-\varrho(t-s)}s^{-\alpha}(t-s)^{2-2\alpha-\gamma-\kappa+\mu}ds\le c\left\|u\right\|^{(\varrho)}_{\gamma,\delta}\varrho^{3\alpha+\gamma+\kappa-\mu-3}\,,\end{aligned}$$

and for the second,

$$\begin{aligned}c\left\|u\right\|^{(\varrho)}_{\gamma,\delta}&\int_0^t\int_0^\tau e^{-\varrho(t-s)}s^{-\alpha}(\tau-s)^{1-\alpha+\mu}ds(t-\tau)^{-\gamma-\alpha}d\tau\\&\le c\left\|u\right\|^{(\varrho)}_{\gamma,\delta}\varrho^{3\alpha+\gamma+\kappa-\mu-3}.\end{aligned}$$

Note that $\alpha+\kappa<1$ and $\alpha-\nu<1$.
Considering the third term with $j(s,t)-j(s,\tau)$ inserted, we proceed as before and use Fubini to get the bound

$$\begin{aligned}c&\int_0^\tau\int_\tau^t s^{-\alpha}(t-s)^{-\kappa}\left\|u(s)\right\|_\delta(\theta-s)^{\mu-1}d\theta ds\\&\le c\int_0^\tau s^{-\alpha}(\tau-s)^{-\kappa-\nu}\left\|u(s)\right\|_\delta\int_\tau^t(\theta-s)^{\nu+\mu-1}d\theta ds\\&\le c\left\|u\right\|^{(\varrho)}_{\gamma,\delta}\int_0^\tau e^{\varrho s}s^{-\alpha}(\tau-s)^{-\kappa-\nu}ds(t-\tau)^{\nu+\mu}\,,\end{aligned} \tag{5.24}$$

$\nu>\gamma$, close to γ. Note that $0<\mu+\nu<1$. Integrating, we observe the upper estimate

$$\begin{aligned}c\left\|u\right\|^{(\varrho)}_{\gamma,\delta}&\int_0^\tau e^{-\varrho(t-s)}s^{-\alpha}(\tau-s)^{-\kappa-\nu}ds\int_0^t(t-\tau)^{\nu-\gamma+\mu-1}d\tau\\&\le c\left\|u\right\|^{(\varrho)}_{\gamma,\delta}\varrho^{\alpha+\kappa+\nu-1}\,.\end{aligned}$$

This shows that also

$$e^{-\varrho t}\int_0^t \frac{\|J_3(t)-J_3(\tau)\|_\delta}{(t-\tau)^{\gamma+1}}d\tau \le C_3(\varrho)\,\|u\|^{(\varrho)}_{\gamma,\delta}\,,$$

$C_3(\varrho)$ tending to zero as ϱ goes to infinity, what completes the proof. □

Next, we comment on the proof of Proposition 5.1.1.

Proof. It is similar, but simpler: Use

$$\begin{aligned}\left\|\frac{\partial}{\partial y_l}D^{1-\alpha}_{t-}Z_t(s)|H_2^{-\beta}(\mathbb{R}^n,\mathbb{R}^k)\right\| &\le c\left\|D^{1-\alpha}_{t-}Z_t(s)|H_2^{1-\beta}(\mathbb{R}^n,\mathbb{R}^k)\right\| \\ &\le c(t-s)^\mu\left\|Z|C^{1-\alpha'}([0,T],H_2^{1-\beta}(\mathbb{R}^n,\mathbb{R}^k))\right\|.\end{aligned}$$

and follow the pattern of the preceding proof. Now J_2 vanishes, and for J_1 and J_3 we can modify the former estimates in an obvious way:
First split J_1 according to (5.13). For the summand corresponding to the last one there, (5.14) yields the bound $c(t-\tau)^\nu$ with some $\nu>\gamma$. (5.15) and (5.16) yield $c(t-\tau)^{1-\alpha}$ for the middle summand with boundary terms b inserted. Recall that $\gamma<1-\alpha$. With the integral terms j, we can use (5.17), note that

$$(t-\tau)^{1-\alpha-\kappa+\mu}\le cT^{1-\gamma-\alpha-\kappa+\mu}(t-\tau)^\gamma$$

where $\kappa=(\delta+\beta)/2$ and $\gamma+\alpha+\kappa-\mu<1$. The first summand is covered by a bound of type (5.18). Hence

$$\|J_1(t)-J_1(\tau)\|_\delta\le c(t-\tau)^\gamma\quad,\quad 0\le\tau<t\le T\ .$$

Next, turn to J_3, splitted according to (5.20). For the first and second summand in question, we use (5.21) and (5.22) to observe factors $c(t-\tau)^\nu$, $\nu>\gamma$. (5.23) covers the third summand with boundary terms b delivering a factor $c(t-\tau)^{1-\alpha}$. (5.24) contributes a factor $(t-\tau)^{\nu+\mu}$ for the summand with integral terms j. Consequently

$$\|J_3(t)-J_3(\tau)\|_\delta\le c(t-\tau)^\gamma\quad,\quad 0\le\tau<t\le T\ ,$$

what finishes the proof. □

Some additional arguments are required for Proposition 5.1.3.

Proof. Fix $l = 1, ..., n$. Recall that $T_{G_l} u := G_l(u) = (G_l^1(u), ..., G_l^k(u))$.

Step 1: Estimates on the non-linearities. Looking at the equivalent norm (2.20), we deduce that

$$\|T_{G_l} u\|_\delta \le c \|u\|_\delta \ , \tag{5.25}$$

see also [75] Theorem 5.5.1/1. For $\|\cdot\|_\infty$ in place of $\|\cdot\|_\delta$ a similar assertion is obvious.
Next we consider the Fréchet derivative T'_{G_l} of the operator $T_{G_l} : H^\delta_{2,\infty}(\mathbb{R}^n, \mathbb{R}^k) \to H^\delta_{2,\infty}(\mathbb{R}^n, \mathbb{R}^k)$. For any $u, v \in H^\delta_{2,\infty}(\mathbb{R}^n, \mathbb{R}^k)$ it is given by

$$T'_{G_l}(u)v = \mathcal{D}G_l(u)v \ . \tag{5.26}$$

For fixed $x \in \mathbb{R}^n$, $\mathcal{D}G_l(u(x)) \in \mathcal{M}(k \times k, \mathbb{R})$, $v(x) \in \mathbb{R}^k$, and (5.26) is understood in the usual sense of matrix multiplication. For $k = 1$, the proof of (5.26) is given in [75], Theorem 5.5.3/1. As we allow $k \ge 1$, we sketch the argments for convenience: For fixed $x \in \mathbb{R}^n$, Taylor expansion yields

$$\begin{aligned} &G_l(u+v)(x) - G_l(u)(x) - \mathcal{D}G_l(u(x))v(x) \\ &\qquad = \int_0^1 (1-\theta) \left(\mathcal{D}^2 G_l(u(x) - \theta v(x))v(x)\right) v(x) d\theta \ . \end{aligned} \tag{5.27}$$

Now given $z \in \mathbb{R}^k$,

$$B_z(\xi, \eta) := \left(\mathcal{D}^2 G_l(z)\xi - \mathcal{D}^2 G_l(0)\xi\right) \eta \ \ , \ \xi, \eta \in \mathbb{R}^k \ ,$$

defines some $B_z \in L(\mathbb{R}^k, L(\mathbb{R}^k, \mathbb{R}^k))$, which may be seen as bilinear mapping $B_z : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$. We observe that for $h, v, w \in H^\delta_{2,\infty}(\mathbb{R}^n, \mathbb{R}^k)$,

$$\begin{aligned} &B_{h(x)}(w(x), v(x)) - B_{h(y)}(w(y), v(y)) \\ &\qquad = B_{h(x)}(w(x) - w(y), v(x)) + B_{h(x)}(w(y), v(x) - v(y)) \\ &\qquad\qquad + \left(B_{h(x)} - B_{h(y)}\right)(w(y), v(y)) \ . \end{aligned}$$

consequently by the hypotheses on $\mathcal{D}^2 G_l$,

$$\begin{aligned} &\left(B_{h(x)}(w(x), v(x)) - B_{h(y)}(w(y), v(y))\right)^2 \\ &\qquad \le c \left(B_{h(x)}(w(x) - w(y), v(x))\right)^2 + c \left(B_{h(x)}(w(y), v(x) - v(y))\right)^2 \\ &\qquad\qquad + c \left(\left(B_{h(x)} - B_{h(y)}\right)(w(y), v(y))\right)^2 \\ &\qquad \le cM^2 \|v\|^2_\infty (w(x) - w(y))^2 + cM^2 \|w\|^2_\infty (v(x) - v(y))^2 \\ &\qquad\qquad + cL^2 (h(x) - h(y))^2 \|w\|^2_\infty \|v\|^2_\infty \ . \end{aligned}$$

Insert this into the second summand (difference part) of the norm (2.20) with $p = 2$ and note that the $\|\cdot\|_0$-part of this norm can be estimated in a similar way. We obtain that

$$\|B_h(w,v)\|_{\delta,\infty} \leq cM \|w\|_{\delta,\infty} \|v\|_{\delta,\infty} + cL \|w\|_{\delta,\infty} \|v\|_{\delta,\infty} \|h\|_{\delta,\infty} \ .$$

This implies that in $\|\cdot\|_{\delta,\infty}$, the right hand side of (5.27) is bounded by

$$c(1 + \|u\|_{\delta,\infty} + \|v\|_{\delta,\infty}) \|v\|_{\delta,\infty}^2 \ .$$

Hence,

$$\lim_{\|v\|_{\delta,\infty}\to 0} \frac{\|G_l(u+v) - G_l(u) - \mathcal{D}G_l(u)v\|_{\delta,\infty}}{\|v\|_{\delta,\infty}} = 0 \ ,$$

what proves (5.26).
Further, we have

$$\left\|T'_{G_l}(u)v\right\|_{\delta,\infty} \leq c(\|u\|_{\delta,\infty} + 1) \|v\|_{\delta,\infty} \ . \tag{5.28}$$

To see this, note first that

$$\mathcal{D}G_l(u)v = \left(\sum_{j=1}^{k} \frac{\partial G_l^j}{\partial x_1}(u)v_j, ..., \sum_{j=1}^{k} \frac{\partial G_l^j}{\partial x_k}(u)v_j \right) \ . \tag{5.29}$$

For a moment, abuse notation and let $\|\cdot\|_{\delta,\infty}$ also denote the norm in $H^\delta_{2,\infty}(\mathbb{R}^n)$ (that is $k = 1$). Since $H^\delta_{2,\infty}(\mathbb{R}^n)$ is a multiplication algebra and due to (2.20), we have for any $i, j = 1, ..., k$,

$$\begin{aligned}
\left\| \frac{\partial G_l^j}{\partial x_i}(u)v_j \right\|_{\delta,\infty} &\leq \left\| \frac{\partial G_l^j}{\partial x_i}(u) - \frac{\partial G_l^j}{\partial x_i}(0) \right\|_{\delta,\infty} \|v_j\|_{\delta,\infty} + \left\| \frac{\partial G_l^j}{\partial x_i}(0) \right\|_{\delta,\infty} \|v_j\|_{\delta,\infty} \\
&\leq M_0 \|u\|_{\delta,\infty} \|v_j\|_{\delta,\infty} + c \|v_j\|_{\delta,\infty} \\
&\leq c(\|u\|_{\delta,\infty} + 1) \|v_j\|_{\delta,\infty} \ ,
\end{aligned}$$

where $M_0 = \sup_{x\in\mathbb{R}^k} \left\| \mathcal{D}\left(\frac{\partial G_l^j}{\partial x_i} - \frac{\partial G_l^j}{\partial x_i}(0) \right)(x) \right\|_{L(\mathbb{R}^k,\mathbb{R}^k)}$. Summing over j according to (5.29) this implies (5.28).
By the mean value theorem it now follows that

$$\begin{aligned}
\|T_{G_l}u - T_{G_l}v\|_\delta &\leq \int_0^1 \left\|T'_{G_l}(\theta u + (1-\theta)v)(u-v)\right\|_\delta d\theta \\
&\leq c \|u - v\|_{\delta,\infty} (\|u\|_{\delta,\infty} + \|v\|_{\delta,\infty} + 1) \ .
\end{aligned} \tag{5.30}$$

Given $u_1, u_2, v_1, v_2 \in H^\delta_\infty(\mathbb{R}^n, \mathbb{R}^k)$, we similarly have

$$T_{G_l}u_1 - T_{G_l}v_1 - T_{G_l}u_2 + T_{G_l}v_2 = \int_0^1 T'_{G_l}(\theta u_1 + (1-\theta)v_1)(u_1 - v_1 - u_2 + v_2)d\theta$$
$$+ \int_0^1 \left(T'_{G_l}(\theta u_1 - (1-\theta)v_1) - T'_{G_l}(\theta u_2 - (1-\theta)v_2)\right)(u_2 - v_2)d\theta \ ,$$

and taking the $\|\cdot\|_\delta$-norm, another application of the mean value theorem together with the Lipschitz property of $\mathcal{D}G_l$ (due to the boundedness of $\mathcal{D}^2G_l$) yields the bound

$$c\,\|u_1 - v_1 - u_2 + v_2\|_{\delta,\infty}\Big(\|u_1\|_{\delta,\infty} + \|v_1\|_{\delta,\infty} + 1\Big)$$
$$+ c\,\|u_2 - v_2\|_{\delta,\infty}\Big(\|u_1\|_{\delta,\infty} + \|v_1\|_{\delta,\infty} + \|u_2\|_{\delta,\infty} + \|v_2\|_{\delta,\infty} + 1\Big) \ , \quad (5.31)$$

cf. [56].

Step 2: An invariant subset. We show that for $\varrho_0 \geq 1$ large enough, the integral operator (5.3) maps the closed ball

$$B^{(\varrho_0)}(0, e^{-\varrho_0 T}) := \left\{u \in W^\gamma([0,T], \mathring{H}^\delta_{2,\infty}(\mathbb{R}^n, \mathbb{R}^k)) : \|u\|^{(\varrho_0)}_{\gamma,\delta,\infty} \leq e^{-\varrho_0 T}\right\}$$

into itself.
We follow the proof of Proposition 5.1.2. Given $u \in W^\gamma([0,T], \mathring{H}^\delta_{2,\infty}(\mathbb{R}^n, \mathbb{R}^k))$, fix $l = 1, ..., n$ and denote by $J_1(t,u)$, $J_2(t,u)$, $J_3(t,u)$ the single summands on the right hand side of the corresponding special case of representation (5.5).
Using (5.25), the estimates involving $\|J_i(t,u)\|_\delta$ and $\|J_i(t,u) - J_i(\tau,u)\|_\delta$, $i = 1,3$, $0 \leq \tau < t \leq T$, carry over from that proof, leading to bounds of type

$$c\,\|u\|^{(\varrho_0)}_{\gamma,\delta,\infty}\,\varrho_0^{\alpha+\kappa+\nu-1} \ , \ c\,\|u\|^{(\varrho_0)}_{\gamma,\delta,\infty}\,\varrho_0^{\alpha+\kappa+\gamma-1} \ , \ c\,\|u\|^{(\varrho_0)}_{\gamma,\delta,\infty}\,\varrho_0^{\alpha+\kappa+\gamma-\mu-1} \ ,$$
$$\text{or} \quad c\,\|u\|^{(\varrho_0)}_{\gamma,\delta,\infty}\,\varrho_0^{3\alpha+\gamma+\kappa-\mu-3} \ , \quad (5.32)$$

where $\kappa = (\delta+\beta)/2$ and with α, ν slightly bigger than γ as specified there. Now recall that

$$\|P(t)u\|_\infty \leq ct^{-n/4}\,\|u\|_0 \quad , \quad t > 0 \ ,$$

here $\|\cdot\|_0$ denotes the norm in $L_2(D,\mathbb{R}^k)$, see e.g. [8].
By isomorphic lifting, this yields estimates on the corresponding terms involving $\|J_i(t,u)\|_\infty$ and $\|J_i(t,u)-J_i(\tau,u)\|_\infty$, $i=1,3$, which are analogous to those written in (5.32), but with κ replaced by $\beta/2+n/4$.
For the terms with $\|J_2(t,u)\|_\delta$ and $\|J_2(t,u)-J_2(\tau,u)\|_\delta$, bounds of type

$$c\,\|u\|^{(\varrho_0)}_{\gamma,\delta,\infty}\,\varrho_0^{\kappa+\nu-1}\ ,\ \ c\,\|u\|^{(\varrho_0)}_{\gamma,\delta,\infty}\,\varrho_0^{2\alpha+\gamma+\kappa-\mu-2}\ \ \text{and}\ \ c\,\|u\|^{(\varrho_0)}_{\gamma,\delta,\infty}\,\varrho_0^{\gamma+\kappa-\mu-1} \tag{5.33}$$

follow, $\nu>\gamma$, but close. Taking the $\|\cdot\|_\infty$-norm instead, the bounds hold with $\beta/2+n/4$ in place of κ.
Since $\|u\|_{\gamma,\delta,\infty}\le 1$ for $u\in B^{(\varrho_0)}(0,e^{-\varrho_0 T})$, it is now sufficient to choose $\varrho_0\ge 1$ large enough to make sure that the images of all u from $B^{(\varrho_0)}(0,e^{-\varrho_0 T})$ have $\|\cdot\|^{\varrho_0}_{\gamma,\delta,\infty}$-norm less than $e^{\varrho_0 T}$.

Step 3: Contractivity. We show that for $\varrho\ge 1$ large enough, (5.3) is a contraction in $B^{(\varrho_0)}(0,e^{-\varrho_0 T})$.
Proceeding as before and using (5.30), we get for instance

$$\begin{aligned} e^{-\varrho t}\,\|J_1(t,u)-J_1(t,v)\|_\delta &\le c\,\|u-v\|^{(\varrho)}_{\gamma,\delta,\infty}\left(\|u\|_{\gamma,\delta,\infty}+\|v\|_{\gamma,\delta,\infty}+1\right)\varrho^{\alpha+\kappa-\nu-1}\\ &\le c\,\|u-v\|^{(\varrho)}_{\gamma,\delta,\infty}\,\varrho^{\alpha+\kappa-\nu-1}\ , \end{aligned}$$

$u,v\in B^{(\varrho_0)}(0,e^{-\varrho_0 T})$, and similary for the other bounds in (5.32). Analogous arguments for $J_3(t,u)-J_3(t,v)$ and $J_i(t,u)-J_i(t,v)-J_i(\tau,u)+J_i(\tau,v)$, $i=1,3$, yield bounds of type $c\,\|u-v\|^{(\varrho)}_{\gamma,\delta,\infty}\,\varrho^{\alpha+\kappa+\nu-1}$, similarly for the other versions in (5.32).
For $\|\cdot\|_\infty$, κ is to be replaced by $\beta/2+n/4$.
On $J_2(t,u)-J_2(t,v)$ and $J_2(t,u)-J_2(t,v)-J_2(\tau,u)+J_2(\tau,v)$, we use (5.31) to arrive at the upper bound $c\,\|u-v\|^{(\varrho)}_{\gamma,\delta,\infty}\,\varrho^{\kappa+\nu-1}$, or one of the other bounds from (5.33).
κ is to be replaced by $\beta/2+n/4$ if $\|\cdot\|_\infty$ is considered.
Now choose $\varrho\ge\varrho_0$ sufficiently large. □

5.2 Correctness of the definition

As a consequence we observe that the integral both in the sense of Proposition 5.1.1 and Proposition 5.1.2 can be rewritten as *forward limit*, similar to the *forward integral* from Section 3.2. For $l = 1, ..., n$, set

$$\partial_{l,r}\varphi(x) := \varphi(x + re_l) - \varphi(x), \quad r > 0 \ ,$$

to denote the *forward difference* of a function φ in direction e_l. As before, let

$$\nabla_r^+\varphi(x) = (\partial_{1,r}\varphi(x), ..., \partial_{n,r}\varphi(x)), \quad r > 0 \ ,$$

denote the *forward pre-gradient* of φ. Now put

$$I_t^\alpha\left(g, \frac{\partial}{\partial t}\nabla_r^+ Z\right) := (-1)^\alpha \int_0^t A^\alpha P(t-s)\left\langle g(s), \nabla_r^+ D_{t-}^{1-\alpha} Z_t(s)\right\rangle ds \qquad (5.34)$$
$$+ c_\alpha(-1)^\alpha \int_0^t \int_0^s (s-\sigma)^{-\alpha-1} P(t-\sigma)\left\langle (g(s) - g(\sigma)), \nabla_r^+ D_{t-}^{1-\alpha} Z_t(s)\right\rangle d\sigma ds$$
$$+ c_\alpha(-1)^\alpha \int_0^t \int_s^\infty \sigma^{-\alpha-1} P(\sigma + t - s)\left\langle g(s), \nabla_r^+ D_{t-}^{1-\alpha} Z_t(s)\right\rangle d\sigma ds \ .$$

for $t > 0$, $r > 0$ and with some $0 < \alpha < 1$.

Corollary 5.2.1. *(i) Under the hypotheses of Proposition 5.1.1, we have*

$$I_t^\alpha\left(G, \frac{\partial}{\partial t}\nabla Z\right) = \lim_{r\to 0} I_t^\alpha\left(G, \frac{\partial}{\partial t}\nabla_r^+ Z\right) \ , \ t > 0 \ ,$$

the limit taken in the strong sense in $\mathring{H}_2^\delta(D, \mathbb{R}^k)$.

(ii) Under the hypotheses of Propositions 5.1.2 and 5.1.3, we have

$$I_t^\alpha\left(G(u), \frac{\partial}{\partial t}\nabla Z\right) = \lim_{r\to 0} I_t^\alpha\left(G(u), \frac{\partial}{\partial t}\nabla_r^+ Z\right) \ , \ t > 0 \ ,$$

for any $u \in W^\gamma([0,T], \mathring{H}_{2(,\infty)}^\delta(D, \mathbb{R}^k))$ *in the strong sense in* $\mathring{H}_{2(,\infty)}^\delta(D, \mathbb{R}^k)$.

A similar assertion is true if the forward differences are replaced by backward differences.

Proof. Assertion (ii) follows applying Lemma 2.3.1: For $v \in H_2^\delta(\mathbb{R}^n)$ and $\varphi \in H_q^{-\beta}(\mathbb{R}^n)$, $q > 2 \vee (n/\delta)$,

$$\left\| v\left(\frac{1}{r}\partial_{l,r}^+\varphi - \frac{\partial}{\partial y_l}\varphi\right)\right\|_{-\beta} \le \|v\|_\delta \left\|\frac{1}{r}\partial_{l,r}^+\varphi - \frac{\partial}{\partial y_l}\varphi | H_q^{-\beta}(\mathbb{R}^n, \mathbb{R}^k)\right\| \ . \qquad (5.35)$$

This tends to zero as r does, since by translation invariance of the $L_q(\mathbb{R}^n)$-norm,

$$\lim_{r\to 0}\left\|\left((1+|\xi|^2)^{-\beta/2}\varphi^\wedge\right)^\vee \circ T_{tre_l} - \left((1+|\xi|^2)^{-\beta/2}\varphi^\wedge\right)^\vee |L_q(\mathbb{R}^n)\right\| = 0$$

for any $t > 0$. Here $\psi \circ T_a(x) = \psi(x+a)$, $a \in \mathbb{R}^n$, denotes the translation, above it is applied in the sense of Schwartz distributions. Assertion (i) follows similarly. □

Next the correctness of Definition 4.1.1 is verified:

Lemma 5.2.1. *Under the hypotheses of Propositions 5.1.1 and interpreted according to (5.1), Definition 4.1.1 is correct, i.e. the existence and the value of the integral do not depend on the particular choice of α:*

$$I_t^\alpha\left(G, \frac{\partial}{\partial t}\nabla Z\right) = I_t\left(G, \frac{\partial}{\partial t}\nabla Z\right) .$$

If the hypotheses of Proposition 5.1.2 respectively 5.1.3 hold, the same is true for the mapping (5.3):

$$I_t^\alpha\left(G(u), \frac{\partial}{\partial t}\nabla Z\right) = I_t\left(G(u), \frac{\partial}{\partial t}\nabla Z\right) .$$

Proof. It suffices to consider the members of (5.34) for fixed $r > 0$.
We consider Definition 4.1.1 interpreted according to (5.3), the case (5.1) is similar.
Given $0 < \alpha, \alpha' < 1$, both satisfying the mentioned assumptions, we show that the integral value remains unchanged if $\alpha' = \alpha + \nu$, $\nu > 0$ replaces α.
In the following, $l = 1, ..., n$ is fixed, $h(s) := D_{t-}^{1-\alpha}\partial_{l,r}^+ Z_t(s)$ is assumed to exist, and $G(s)$ is written to denote $G_l(u(s))$. Note that we use the definition of the fractional integral operator I_{t-}^ν which includes the factor $(-1)^{-\nu}$, see [40] or [91].
By semigroup and invertibility properties of fractional integrals and derivatives, the first summand in (5.5) with G in place of g yields

$$\begin{aligned}(-1)^{\alpha+\nu}\int_0^t A^{\alpha+\nu}P(t-s)\left(G(s)\cdot I_{t-}^\nu h\right)(s)ds \\ = (-1)^\alpha \int_0^t\int_s^t A^{\alpha+\nu}P(t-s)(\tau-s)^{\nu-1}G(s)\cdot h(\tau)\,d\tau ds\ .\end{aligned}$$

Applying Lemma 2.2.1 to the $E = H_2^{-\beta}(\mathbb{R}^n,\mathbb{R}^k)$-valued function $f := G(\cdot)\cdot h(\tau)$, we obtain three terms: The first is

$$(-1)^\alpha \int_0^t I_{0+}^\nu \left(D_{0+}^{\alpha+\nu} P(t-\cdot)G(\cdot)\cdot h(\tau)\right)(\tau)d\tau$$
$$= (-1)^\alpha \int_0^t D_{0+}^\alpha \left(P(t-\cdot)G(\cdot)\cdot h(\tau)\right)(\tau)d\tau\ , \quad (5.36)$$

the second is

$$-(-1)^\alpha c_{\alpha+\nu}\int_0^t I_{0+}^\nu \Psi_{t,\tau}(\tau)d\tau\ ,$$

where

$$\Psi_{t,\tau}(s) = P(t-s)\int_s^\infty u^{-(\alpha+\nu)-1}P(u)G(s)\cdot h(\tau)du\ ,$$

and the third equals

$$-(-1)^\alpha c_{\alpha+\nu}\int_0^t I_{0+}^\nu \Lambda_{t,\tau}(\tau)d\tau\ ,$$

where

$$\Lambda_{t,\tau}(s) = P(t-s)\int_0^s u^{-(\alpha+\nu)-1}(G(s)-G(s-u))\cdot h(\tau)\,du\ .$$

From the second summand in (5.5) we obtain

$$(-1)^{\alpha+\nu}c_{\alpha+\nu}\int_0^t\int_0^s (s-\sigma)^{-(\alpha+\nu)-1}P(t-\sigma)(G(s)-G(\sigma))\cdot I_{t-}^\nu h(s))\,d\sigma ds$$
$$= \frac{(-1)^{\alpha+\nu}c_{\alpha+\nu}}{\Gamma(\nu)}\int_0^t\int_s^t\int_0^s (s-\sigma)^{-(\alpha+\nu)-1}\times$$
$$\times P(t-\sigma)(G(s)-G(\sigma))\cdot h(\tau)d\sigma(\tau-s)^{\nu-1}d\tau ds$$
$$= (-1)^\alpha c_{\alpha+\nu}\int_0^t I_{0+}^\nu \Lambda_{t,\tau}(\tau)\,d\tau\ ,$$

and from the third summand in (5.5),

$$(-1)^{\alpha+\nu}c_{\alpha+\nu}\int_0^t\int_s^\infty u^{-(\alpha+\nu)-1}P(u)P(t-s)(G(s)\cdot I_{t-}^\nu h(s))\,duds$$
$$= (-1)^\alpha c_{\alpha+\nu}\int_0^t I_{0+}^\nu \Psi_{t,\tau}(\tau)\,d\tau\ .$$

The terms cancel and by (5.36) together with Lemma 2.2.1 we arrive at the integral with α according to Definition 4.1.1 and interpretation (5.3). Taking limits as r goes to zero and using Corollary 5.2.1 (in particular, the estimate (5.35)), the values are seen to agree in $H_2^\delta(\mathbb{R}^n, \mathbb{R}^k)$. $\square$

5.3 Conclusion of the main results

Theorem 4.5.1 readily follows from Proposition 5.1.1. Note that by the mapping properties of the semigroup,

$$\frac{\|(P(t-\tau)-I)P(\tau)f\|_\delta}{(t-\tau)^\gamma} \le c(t-\tau)^\varepsilon \|f\|_{2\gamma+\delta+\varepsilon} \quad , \quad 0 \le \tau < t \le T \ .$$

Theorem 4.2.1 follows from Banach's fixed point theorem and a similar observation: We have $\|P(t)f\|_\delta \le ce^{-\mu t} \|f\|_\delta$ and

$$\int_0^t \frac{\|P(t)f - P(\tau)f\|_\delta}{(t-\tau)^{\gamma+1}} d\tau \le c \|f\|_{2\gamma+\delta+\varepsilon} \int_0^t (t-\tau)^{\varepsilon-1} d\tau \ .$$

Theorem 4.3.1 follows similarly, having choosen a common ϱ_0 in Proposition 5.1.3 and in the following Lemma 5.3.1. Note that $\|P(t)f\|_\infty \le t^{-n/4} \|f\|_\delta$ and

$$\int_0^t \frac{\|P(t)f - P(\tau)f\|_\infty}{(t-\tau)^{\gamma+1}} d\tau \le c \|f\|_{2\gamma+\delta+\varepsilon} \int_0^t \tau^{-n/4}(t-\tau)^{\varepsilon-1} d\tau \ .$$

Set $J_0(t,u) := \int_0^t P(t-s)F(u(s))ds$.

Lemma 5.3.1. *For $0 < \gamma, \delta < 1$ such that $\gamma + n/4 < 1$ and $\varrho_0 \ge 1$ large enough, $u \mapsto J_0(\cdot, u)$ maps the closed ball $B^{(\varrho_0)}(0, e^{-\varrho_0 T})$ into itself and for $\varrho \ge \varrho_0$,*

$$\|J_0(\cdot,u) - J_0(\cdot,v)\|^{(\varrho)}_{\gamma,\delta,\infty} \le C(\varrho) \|u-v\|^{(\varrho)}_{\gamma,\delta,\infty} ,$$

$u, v \in B^{(\varrho_0)}(0, e^{-\varrho_0 T})$, where $C(\varrho)$ tends to zero as ϱ tends to infinity.

Proof. The first assertion is seen as follows. We have

$$e^{-\varrho t} \|J_0(t,u)\|_\delta \le ce^{-\varrho t} \int_0^t \|T_F u(s)\|_\delta \, ds \le c \|u\|^{(\varrho)}_{\gamma,\delta,\infty} \varrho^{-1} \ .$$

Replacing $\|\cdot\|_\delta$ by $\|\cdot\|_\infty$, we arrive at $c\,\|u\|^{(\varrho)}_{\gamma,\delta,\infty}\,\varrho^{n/4-1}$. For $0 \le \tau < t \le T$, use

$$J_0(t,u)-J_0(\tau,u) = \int_\tau^t P(t-s)F(u(s))ds + \int_0^\tau [P(t-\tau)-I]P(\tau-s)F(u(s))ds$$

to arrive at bounds $c\,\|u\|^{(\varrho)}_{\gamma,\delta}\,\varrho^{\gamma-1}$ for the norm $\|\cdot\|_\delta$, and $c\,\|u\|^{(\varrho)}_{\gamma,\delta}\,\varrho^{\gamma+n/4-1}$ for $\|\cdot\|_\infty$. To deduce the second assertion, use (5.30) to get

$$\begin{aligned}
\|J_0(t,u) - J_0(\tau,v)\|_\delta &\le c\int_0^t \|T_F u(s) - T_F v(s)\|_\delta\, ds \\
&\le c\int_0^t \|u(s)-v(s)\|_{\delta,\infty}\Big(\|u(s)\|_{\delta,\infty} + \|v(s)\|_{\delta,\infty} + 1\Big)\,ds\ .
\end{aligned}$$

We arrive at the upper bound $c\,\|u-v\|^{(\varrho)}_{\gamma,\delta,\infty}\Big(\|u\|_{\gamma,\delta,\infty} + \|v\|_{\gamma,\delta,\infty} + 1\Big)\,\varrho^{-1}$, for $\|\cdot\|_\infty$, ϱ^{-1} is to be replaced by $\varrho^{n/4-1}$. For the term contributed by $J_0(t,u)-J_0(t,v)-J_0(\tau,u)-J_0(\tau,v)$, we obtain the estimate $c\,\|u-v\|^{(\varrho)}_{\gamma,\delta,\infty}\,(\|u\|_{\gamma,\delta,\infty} + \|v\|_{\gamma,\delta,\infty}+1)\varrho^{\gamma-1}$ for the $\|\cdot\|_\delta$-part, for the $\|\cdot\|_\infty$-part replace $\varrho^{\gamma-1}$ by $\varrho^{\gamma+n/4-1}$. Now note that $\|u\|_{\gamma,\delta,\infty} + \|v\|_{\gamma,\delta,\infty} \le 2$, and choose $\varrho \ge \varrho_0$ large enough. $\square$

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Hiermit erkläre ich,

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Bei der Auswahl und Auswertung des Materials sowie bei der Herstellung des Manuskripts haben mich folgende Personen unterstützt: Prof. Dr. Martina Zähle, Jena.

Ich habe die gleiche, in wesentlichen Teilen ähnliche bzw. eine andere Abhandlung noch bei keiner anderen Hochschule als Dissertation eingereicht.

Jena, den 25.11.2008 Michael Hinz